# A Relationship between Risk and Time Preferences *

Kota SAITO †
California Institute of Technology

April 23, 2015

**Abstract**

This paper investigates a general relationship between risk and time preferences. I consider a decision maker who chooses between consumption of a particular prize in one period and a different prize in another period. The individual believes that today's good is certain, and that, as the promised date for a future good becomes increasingly distant, the probability of consuming the good decreases. Under these assumptions, this paper shows that the individuals exhibits the common ratio effect, the certainty effect, and is an expected utility maximizer if and only if he discounts hyperbolically, quasi-hyperbolically and exponentially, respectively.

Keywords: Allais paradox; hyperbolic discounting.

## 1 Introduction

Conventional wisdom recognizes that the future is uncertain in many respects. Several researchers, therefore, have claimed that there should be a relationship between risk and

*This paper subsumes Saito, K. (2009) "A Relationship between Risk and Time Preferences." The Center for Mathematical Studies in Economics & Management Sciences, Northwestern University, Discussion Paper No. 1477.

†I appreciate Thomas Epper, Daniele Pennesi, Charlie Sprenger, and Pietro Ortoleva for helpful discussions and comments. I also wish to thank Federico Echenique, Taisuke Imai, Gerelt Tserenjigmid. I acknowledge financial support from the NSF grant SES 0820333 and the Center for Economic Theory of the Economics Department of Northwestern University.

time preferences. Many experiments such as Keren and Roelofsma (1995), Weber and Chapman (2005), and Epper, Fehr-Duda, and Bruhin (2011) have suggested such a relationship.[1]

Based on this intuition and experimental evidence, several studies have tried to explain future discounting on the basis of the uncertainty associated with future payoffs. In particular, those studies have tried to capture hyperbolic discounting or quasi-hyperbolic discounting because these discounting imply dynamically inconsistent preferences. For example, Dasgupta and Maskin (2005) assume that it is uncertain when a decision maker can consume the future goods. Sozou (1998) assume that it is uncertain whether a decision maker can consume the future goods or not.[2] These studies assume that the decision makers are expected utility maximizers. On the other hand, Halevy (2008) and Epper and Fehr-Duda (2015) assume that the decision maker is a rank dependent utility maximizer.

It is difficult, however, to identify the fundamental relationships between risk and time preferences, in particular, dynamic inconsistent preferences for three reasons. First, each paper uses different utility functions, probability-weighting functions, and probability distributions capturing the future uncertainty. Moreover, there is no consensus about the assumptions on utility functions. (See the debate between Andreoni and Sprenger (2012, forthcoming) and Epper and Fehr-Duda (forthcoming).)

Second, in most of the conventional studies, it is not clear whether they are studying static choices (i.e, choices at a fixed time) or dynamic choices (i.e., choices with variable times). Moreover, some papers study only static choices. As Dasgupta and Maskin (2005) point out, there are two distinct meanings for the term "hyperbolic discounting" in the literature. One applies to dynamic choices and implies dynamic inconsistency. The other applies to static choices and does not imply dynamic inconsistency. Most of economists, Laibson (1997), O'Donoghue and Rabin (1999), and Dasgupta and Maskin (2005) for example, are interested in the dynamic concept. In this paper, we explicitly study dynamic choices.

Finally, conventional studies have focused only on one way relationship from risk preferences to intertemporal preferences. The other direction of relationships from intertemporal preferences to risk preferences has not been studied extensively.

[1] There are a few experiments which could not find such correlation. See Dean and Ortoleva (2015).

[2] Sozou (1998), moreover, assume that the hazard rate of the probability of missing the future goods is constant but unknown to the decision maker.

The purpose of the present paper is to establish general relationships between risk and dynamic time preferences, without assuming specific forms of utility functions and probability distributions. To achieve this purpose as simply as possible, I consider a decision maker who chooses between consumption of a particular prize in one period and a different prize in another period. The individual discounts a future good because it is uncertain whether he can consume it or not. I assume a weak condition on the probability function representing the uncertainty: that today's good is certain, but as the promised date for a future good becomes increasingly distant, the probability of consuming the good decreases to zero. I call the above property *regularity*. For example, if the probability reflects the decision maker's subjective mortality rate, then the regularity condition would be reasonable.

I consider two different time setups. When the set of time periods is continuous (i.e., non-negative real numbers), I show the following: (i) a decision maker exhibits a type of Allais's paradox, the common ratio effect, if and only if he discounts hyperbolically; (ii) he exhibits a special case of the common ratio effect, the certainty effect, if and only if he discounts quasi-hyperbolically; and (iii) he is an expected utility maximizer if and only if he is temporally unbiased (an exponential discounter).

In addition, I consider the case in which the set of time periods is discrete (i.e., non-negative integers). Then, I show the following: (i') if a decision maker exhibits the common ratio effect, then he discounts hyperbolically; (ii') if he exhibits the certainty effect, then he discounts quasi-hyperbolically; and (iii') if he is an expected utility maximizer, then he is temporally unbiased.

Since most of the existing research satisfies the regularity condition, the results of the present paper are consistent with the results obtained in that research. For example, in a continuous time setup, Epper and Fehr-Duda (2015) show that the subproportionality of probability weighting function, which captures the common ratio effect, implies hyperbolic discounting. This result is consistent with part (i) of my theorem. In a discrete time setup, Halevy (2008) shows that the common ratio effect implies a property of quasi-hyperbolic discounting, Diminishing Impatience.[3] In the same setup, Saito (2011) shows that a special case of the common-ratio effect, the certainty effect, is enough to imply Diminishing Impatience. These results are consistent with part (ii') of my theorem.[4]

[3] Halevy (2008) also claims the converse of the statement, which is incorrect as shown by Saito (2011).

[4] Saito (2011, Claim 3) also claims the converse of the statement, which is incorrect as shown by

Although, parts (iii) and (iii') of my theorem may seem like trivial results given the other parts, parts (iii) and (iii') have an important implication: the approach of "explaining" dynamic inconsistency by assuming non-constant hazard function together with a standard expected utility model, as is currently prevalent in psychology and biology, cannot succeed. For example, Kagel et al. (1986) and Green and Myerson (1996) argue that the decreasing rate of the Poisson hazard rate over time leads to hyperbolic discounting. Sozou (1998) offers an alternative theory in which the hazard rate is constant but unknown to the decision maker. However, part (iii) and (iii') of the theorem show that this approach must lead to temporally unbiased preferences, i.e., to dynamic consistency. That is because the probability (survival) function defined from the hazard function satisfies the regularity condition of this paper. In fact, most researchers who adopt the hazard function approach describe a preference reversal in *static* choices, but not in dynamic choices.

Moreover, the results of the present paper are consistent with an experimental studies. Parts (i) and (i') of my theorem are consistent with an experimental study by Epper, Fehr-Duda, and Bruhin (2011). Given their experiments, Epper, Fehr-Duda, and Bruhin (2011) estimate the nonlinearity of subjects' probability-weighting function, which captures the common ratio effect, and the degree of decreasing their discount rates, which captures hyperbolic discounting. Epper, Fehr-Duda, and Bruhin (2011) found a strong correlation between the two estimates.

Parts (ii) and (ii') of my theorem are also consistent with experimental studies such as Weber and Chapman (2005) and Andreoni and Sprenger (2012). These studies suggest that the driving force of dynamic inconsistency is not the common ratio effect but its special case, the certainty effect. Parts (ii) and (ii') are formalizations of such a conjecture.[5]

Finally, I mention a very recent result. Pennesi (2015) has proposed an interesting way to capture hyperbolic discounting, which is not incorporated by my results in this paper. Pennesi's (2015) model is unique in that the uncertainty about the future payoffs is completely subjective in his model.

The rest of the paper is organized as follows. Section 2 formally defines preferences exhibiting the Allais paradox. Section 3 defines preferences exhibiting hyperbolic and

Chakraborty and Halevy (2015).

[5]In the experiments by Weber and Chapman (2005) and Andreoni and Sprenger (2012), subjects made static choices, not dynamic choice.

quasi-hyperbolic discounting. Section 4 shows the theorem.

## 2 The Allais Paradox

In this section, I consider a risk preference $\succsim^r$ on the set of binary lotteries, defined as follows:

$$\Delta = \Big\{(x, p; 0, 1-p) \;\big|\; x \in X \text{ and } p \in [0,1]\Big\},$$

where $X$ is a non-degenerate closed interval on $\mathbb{R}$ including 0. I formally define the common ratio effect and the certainty effect on the preference $\succsim^r$, which are typical effects of the Allais paradox. The common ratio effect is characterized as follows: Suppose that subjects choose either a safer option which gives a smaller gain $x$ with a higher probability $\eta$, or a riskier option which gives a larger gain $y$ with a lower probability $\eta\mu$, where $\mu < 1$. As $\eta$ falls, subjects switch their choice from the safe option to the risky option. Note that for both options, reducing $\eta$ means increasing the risk of getting nothing. Formally, the common ratio effect is defined as follows:

Definition: $\succsim^r$ is said to exhibit *the common ratio effect*[6] if, for any $x, y \in X$ and $\mu, \tilde{\eta} \in (0,1]$ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$,

$$(x, \eta) \prec^r (y, \eta\mu) \text{ for all } \eta \in (0, \tilde{\eta}) \text{ and } (x, \eta) \succ^r (y, \eta\mu) \text{ for all } \eta \in (\tilde{\eta}, 1].$$

This definition appears in Starmer (2000, p. 337). The general definition provided by Machina (1982, p. 305) also becomes equivalent to the above definition within the set of simple binary lotteries. This tendency is called the certainty effect specifically in regard to the choice between a sure option and a risky option. So the condition characterizing the certainty effect is the special case of the common ratio effect, when $\tilde{\eta} = 1$:

Definition: $\succsim^r$ is said to exhibit *the certainty effect* if, for any $x, y \in X$ and $\mu \in [0,1)$ such that $(x, 1) \sim^r (y, \mu)$,

$$(x, \eta) \prec^r (y, \eta\mu) \text{ for all } \eta \in (0, 1).$$

[6]Under the standard assumption of monotonicity and continuity axioms, for any $x, y \in X$ and $\tilde{\eta} \in [0,1]$, there exists $\mu$ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$. So the condition cannot be satisfied by any trivial way.

By definition, if a decision maker exhibits the common ratio effect, then he exhibits the certainty effect.[7]

Finally, in the set $\Delta$ of binary lotteries, the independence axiom reduces to the following:

DEFINITION: $\succsim^r$ is said to satisfy *the independence axiom* if, for any $x, y \in X$ and $\mu, \eta, \eta' \in [0,1]$,

$$(x,\eta) \succsim^r (y, \eta\mu) \Leftrightarrow (x, \eta') \succsim^r (y, \eta'\mu).$$

# 3 The Present Bias

In this section, I define how to derive time preferences from risk preferences; I also define preferences exhibiting hyperbolic and quasi-hyperbolic discounting. Consider a decision maker who chooses between consumption of a particular prize in one period and a different prize in a different period. The individual discounts the future goods because it is uncertain whether or not he can consume it. I denote the set of time periods by $T$. In the following, I consider two cases in which $T = \mathbb{R}_+$ (i.e., non-negative real numbers) or $T = \mathbb{Z}_+$ (i.e., non-negative integers).

To capture the uncertainty, let $p(t)$ be the probability that the decision maker can consume the good at a promised time $t \in T$. One interpretation of $p(t)$ corresponds to the probability that the decision maker is alive at time $t$. Another interpretation of $p(t)$, as seen in biology and psychology, is the probability that the goods have not been stolen by other animals by time $t$ (see, for example, Kagel et al. (1986)). These two interpretations are representative of most of the research ascribing future discounting to future uncertainty.

For each $d \in T$, I denote the set of one-time consumptions after time $d$ by $X(d) = \{[x,t] \mid x \in X \text{ and } t \in T \text{ such that } t \geq d\}$. Consider the decision maker's time preference $\succsim_0$ at time 0. The preference $\succsim_0$ is on $X(0)$. Suppose that the decision maker is still alive at date $d \geq 0$. Then the probability that he is still alive and able to consume the good at date $t \geq d$ is the conditional probability $p(t|d) = p(t)/p(d)$. Therefore, the decision

[7]Several experimental studies on the common ratio effect and the certainty effect have found that the preference is reversed by changing the prizes from gains into losses (see, for example, Kahneman and Tversky (1979, p.268)). I can define these preferences by just switching strict preferences from $\succ$ to $\prec$,

Figure 1: Time Structure

maker at time $d$ prefers prize $x$ at time $t$, denoted by $[x, t]$, to another future payoff $[y, s]$ if and only if he prefers the binary lottery $(x, p(t|d))$, which gives $x$ with the probability $p(t|d)$, to the lottery $(y, p(s|d))$. Thus, the decision maker's time preferences $\{\succsim_d\}_{d \in T}$ for each decision time $d \in T$, is defined as follows:

DEFINITION: For all $d \in T$ and $[x, t], [y, s] \in X(d)$,

$$[x, t] \succsim_d [y, s] \Leftrightarrow \big(x, p(t|d)\big) \succsim^r \big(y, p(s|d)\big).$$

Henceforth, I will denote these time preferences by $\{\succsim_d\}$. I am now in a position to define preferences exhibiting hyperbolic and quasi-hyperbolic discounting. Hyperbolic discounting is characterized as follows: Suppose that subjects choose either an earlier, smaller payoff which gives a payoff $x$ at a date $t$ or a later, larger payoff which gives a payoff $y$ at a date $s$, where $x < y$ and $t < s$. At first, many subjects want to wait for the later, larger payoff, that is, they prefer $[y, s]$ to $[x, t]$. After some time $\tilde{d}$, however, they do not want to wait any longer, and consequently reverse their preferences as described in Figure 2.[8]

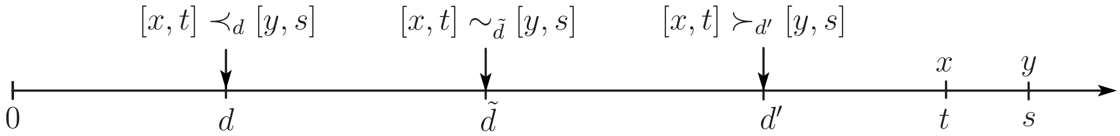

Figure 2: Preferences Exhibiting Hyperbolic Discounting

Hyperbolic discounting is therefore defined as follows:

DEFINITION: $\{\succsim_d\}$ is said to exhibit *hyperbolic* discounting if, for any $x, y \in X$ and

and vice versa. Henceforth, I will mention the case of negative payoffs only in footnotes.

[8]In the following three definitions of time preferences, I focus on positive payoffs for simplicity. For the case of negative payoffs, present biasness appear as procrastination and is defined by the same way just by switching strict preference from $\succ$ to $\prec$, and vice versa. O'Donoghue and Rabin (1999) offers examples of procrastination.

$\tilde{d}, t, s \in T$ such that $[x,t] \sim_{\tilde{d}} [y,s]$ and $\tilde{d} \leq t \leq s$,

$$[x,t] \prec_d [y,s] \text{ for all } d < \tilde{d} \text{ and } [x,t] \succ_d [y,s] \text{ for all } d \text{ such that } t > d > \tilde{d}.$$

Note that the characterization of hyperbolic discounting in Proposition 1 of Dasgupta and Maskin (2005, p. 1293) is exactly the same as above.[9]

Quasi-hyperbolic discounting focuses on present-biased behavior specifically when the promised date for the payoff is close at hand. (See Laibson (1997), for example.) Hence, the condition characterizing quasi-hyperbolic discounting is defined as a special case of hyperbolic discounting, where $\tilde{d} = t$ as follows:

Definition: $\{\succsim_d\}$ is said to exhibit *quasi-hyperbolic* discounting if, for any $x, y \in X$ and $t, s \in T$ such that $[x,t] \sim_t [y,s]$ and $t \leq s$,

$$[x,t] \prec_d [y,s] \text{ for all } d < t.$$

By definition, if a decision maker exhibits hyperbolic discounting, then he exhibits quasi-hyperbolic discounting, but the converse is not true.

Finally, I define *temporally unbiased* preferences, which corresponds to exponential discounting, as follows:

Definition: $\{\succsim_d\}$ is said to be *temporally unbiased* if, for any $x, y \in X$ and $d, d', s, t \in \mathbb{R}_+$,

$$[x,t] \succsim_d [y,s] \Leftrightarrow [x,t] \succsim_{d'} [y,s].$$

# 4 The Theorem

To establish the result of the present paper, I assume a regularity condition on future uncertainty which means that today's good is certain, but, as the promised date for future

[9] In Proposition 1 of Dasgupta and Maskin (2005, p1293), they characterize hyperbolic discounting as follows:
"Assume that there exists $t^*(< T)$ at which the DM is indifferent between $P = (V, T)$ and $P' = (V', T')$ with $0 < V < V'$, $T < T'$, $\cdots$. Then the DM prefers $P'$ to $P$ at $t < t^*$ but prefers $P$ to $P'$ at all $t$ such that $t^* < t < T$". In their notation $(V, T)$ is a future payoff $V$ after time $T$. This characterization is the exactly same as the definition of the present paper.

goods becomes increasingly distant, the probability of consuming the good continuously decreases to zero:

Assumption (Regularity): $p(0) = 1$, $p(\infty) = 0$, and $p$ is strictly decreasing. Moreover if the set $T$ of time period is $\mathbb{R}_+$, then $p$ is continuous.

Theorem: *Suppose that Regularity holds.*

*When the set $T$ of time period is $\mathbb{R}_+$, the following three equivalences hold:*[10]

(i) $\succsim^r$ *exhibits the common ratio effect iff* $\{\succsim_d\}$ *exhibits hyperbolic discounting.*

(ii) $\succsim^r$ *exhibits the certainty effect iff* $\{\succsim_d\}$ *exhibits quasi-hyperbolic discounting.*

(iii) $\succsim^r$ *satisfies the independence axiom iff* $\{\succsim_d\}$ *is temporally unbiased.*

*When the set $T$ of time period is $\mathbb{Z}_+$, the following one way implications hold:*

(i') *If* $\succsim^r$ *exhibits the common ratio effect, then* $\{\succsim_d\}$ *exhibits hyperbolic discounting.*

(ii') *If* $\succsim^r$ *exhibits the certainty effect, then* $\{\succsim_d\}$ *exhibits quasi-hyperbolic discounting.*

(iii') *If* $\succsim^r$ *satisfies the independence axiom, then* $\{\succsim_d\}$ *is temporally unbiased.*

Proof of Theorem: We will prove (i) and (i'). The proofs for the rest of the statements are similar and in the appendix.

Step 1: If $\succsim^r$ exhibits the common ratio effect, then $\{\succsim_d\}$ is hyperbolic.

Proof of Step 1: Choose any $x, y \in X$ and $\tilde{d}, t, s \in T$ such that $[x, t] \sim_{\tilde{d}} [y, s]$ and $\tilde{d} \leq t \leq s$. Then by definition, $(x, p(t|\tilde{d})) \sim^r (y, p(s|\tilde{d})) = (y, p(s|t)p(t|\tilde{d}))$. Fix $d < \tilde{d}$ to show $[x, t] \prec_d [y, s]$. Since $p$ is strictly decreasing, $p(d) > p(\tilde{d})$, so that $p(t|d) < p(t|\tilde{d})$. So the common ratio effect implies that $(x, p(t|d)) \prec^r (y, p(s|t)p(t|d)) = (y, p(s|d))$. Then by definition, $[x, t] \prec_d [y, s]$.

Fix $d > \tilde{d}$ to show that $[x, t] \succ_d [y, s]$. Remember that $(x, p(t|\tilde{d})) \sim^r (y, p(s|t)p(t|\tilde{d}))$. Since $p$ is strictly decreasing, $p(d) < p(\tilde{d})$ so that $p(t|d) > p(t|\tilde{d})$. By the common ratio effect, $(x, p(t|d)) \succ^r (y, p(s|t)p(t|d)) = (y, p(s|d))$. Then by definition, $[x, t] \succ_d [y, s]$.

Step 2: If $\{\succsim_d\}$ exhibits hyperbolic discounting, then $\succsim^r$ exhibits the common ratio effect.

Proof of Step 2: Choose any $x, y \in X$ and $\mu, \tilde{\eta} \in [0, 1]$ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$. Fix $\eta \in (0, \tilde{\eta})$ to show $(x, \eta) \prec^r (y, \eta\mu)$. Since $p$ is strictly decreasing and bijection to

[10] In the section above, I have defined the Allais paradox and present bias for positive payoffs. However, as I mentioned in footnote, I can define these concepts for negative payoffs just by switching strict preferences. Hence, the equivalence here also holds for negative payoffs as well.

$[0,1]$, there exist $t$ and $\tilde{d}$ such that $t \geq \tilde{d} > 0$ and $p(t) = \eta$ and $p(\tilde{d}) = \eta/\tilde{\eta}$. Then, $p(t|\tilde{d}) = \tilde{\eta}$. Also, there exists $s$ such that $s \geq t$ and $p(s) = \mu\eta$. Then, $p(s|t) = \mu$. Hence, $(x, p(t|\tilde{d})) \sim^r (y, p(s|t)p(t|\tilde{d})) = (y, p(s|\tilde{d}))$, so that $[x,t] \sim_{\tilde{d}} [y,s]$, by definition. Therefore, if $\{\succsim_d\}$ is hyperbolic, then $[x,t] \prec_0 [y,s]$. So the definition shows that $(x,\eta) = (x, p(t)) \prec^r (y, p(s)) = (y, \eta\mu)$.

Choose $\eta \in (\tilde{\eta}, 1]$ to show that $(x,\eta) \succ^r (y, \eta\mu)$. Remember that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$. Since $p$ is strictly decreasing and bijection to $[0,1]$, there exist $s, t \in T$ such that $s \geq t$, $p(t) = \tilde{\eta}$, and $p(s) = \mu\tilde{\eta}$. Since $\eta > \tilde{\eta}$, there exists $d$ such that $p(d) = \tilde{\eta}/\eta$. Then, $p(s|t) = \mu$ and $p(t|d) = \eta$. Hence, $[x,t] \sim_0 [y,s]$. By hyperbolic discounting, $[x,t] \succ_d [y,s]$. This implies $(x,\eta) = (x, p(t|d)) \succ^r (y, p(s|d)) = (y, p(s|t)p(t|d)) = (y, \mu\eta)$.

Note that only Step 2 depends on the assumption that the set $T$ of time period is $\mathbb{R}_+$. When the set $T$ of time period is $\mathbb{Z}_+$, only Step 1 holds. So I obtain (i'). The proofs of (ii), (iii), (ii') and (iii') are in the appendix. ■

The proof crucially relies on two structural similarities between risky choices and intertemporal choices. One is the similarity that relates safe outcomes to earlier ones, and risky outcomes to later ones. The other is the similarity that relates increasing risk to moving a decision time forward as well as decreasing risks to moving a decision time backward. These similarities have been suggested by Prelec and Loewenstein (1991), although they have not provided a formal argument.

Only when the set $T$ of time period is $\mathbb{R}_+$, I can translate risky choice to intertemporal choice. When the set $T$ of time period is $\mathbb{Z}_+$, I can obtain only one direction from the property of risk preferences to intertemporal preferences. Indeed, Chakraborty and Halevy (2015) show that in the model of Halevy (2008), Diminishing Impatience does not imply the certainty effect.

Saito (2011 Claim 3), which claims the equivalence results (i), (ii), and (iii) in Halevy's (2008) setup, is incorrect. This is because in his proof Saito (2011) assumes continuous time setup, while Halevy (2008) considers the discrete time setup. In the discrete time setup, the one-way results (i.e., (i'), (ii'), and (iii')) are correct.

As explained in the introduction, the theorem of the paper may be viewed as a generalization of most of the existing research on hyperbolic discounting, because most of them adopt Regularity. In other words, to obtain a relation which is not included in the theorem, it is necessary to violate Regularity. As far as I know, Dasgupta and Maskin

(2005) is the unique example of such approach. They assume not only a constant Poisson mortality rate, but also uncertainty regarding the timing of the payoffs. Accordingly they violate Regularity and describe dynamically inconsistent behavior, despite assuming the expected utility.

The theorem may also answer the question as to what causes hyperbolic discounting. I discuss three possible answers which are compatible with the theorem presented here. The first answer is that the Allais paradox causes hyperbolic discounting (see, for example, Halevy (2008) and Epper and Fehr-Duda (2015)). The second answer is that non-regular uncertainty causes it (see, for example, Dasgupta and Maskin (2005)). The third answer is that a third factor may cause both the Allais paradox and hyperbolic discounting; for example, Fudenberg and Levine (2008) claim that temptation caused by either certainty or presentness would be the common factor. The choice among these three must await future research.

# References


[1] Andreoni, J., and C. Sprenger. 2012. "Risk Preferences Are Not Time Preferences, " *American Economic Review*, 102(7), 3357–3376.

[2] Andreoni, J., and C. Sprenger. forthcoming "Risk Preferences Are Not Time Preferences: Reply, " *American Economic Review*.

[3] Benzion, Uri, Amnon Rapoport, and Joseph Yagil. 1989. " Discount Rates Inferred from Decisions: An Experimental Study," *Management Science*, 35(3):270-284.

[4] Chakraborty, Anujit, and Yoram Halevy. 2015. "Strotz Meets Allais: Diminishing Impatience and the Certainty Effect: Reply and Corrigendum," working paper.

[5] Dasgupta, Partha, and Eric Maskin. 2005. "Uncertainty and Hyperbolic Discounting." *American Economic Review*, 95(4): 1290-1299.

[6] Epper, Thomas and Helga Fehr-Duda. 2015. "The Missing Link: Unifying Risk Taking and Time Discounting," working paper.

[7] Epper, Thomas, Helga Fehr-Duda, and Adrian Bruhin. 2011. “Viewing the Future through a Warped Lens: Why Uncertainty Generates Hyperbolic Discounting,” Journal of Risk and Uncertainty, 43(3), 169–203.

[8] Epper, Thomas and Helga Fehr-Duda. forthcoming. “Balancing on a Budget Line: Comment on Andreoni and Sprenger (2012) ’ s “Risk Preferences Are Not Time Preferences”,” *American Economic Review.*

[9] Fudenberg, Drew, and David Levine. 2008, “Self-control, Risk Aversion, and the Allais Paradox.” Washington University in St. Louis Working paper.

[10] Green, Leonard, and Joel Myerson. 1996. “Exponential Versus Hyperbolic Discounting of Delayed Outcomes: Risk and Waiting Time.” *American Zoologist*, 36: 496-505.

[11] Halevy, Yoram. 2008. “Stroz meets Allais: Diminishing Impatience and the Certainty Effect.” *American Economic Review*, 98(3): 1145-62.

[12] Kagel, H. John, Green Leonard, and Thomas Caraco. 1986. “When Foragers Discount The Future: Constraint or Adaptation?” *Animal Behavior*, 34(1): 271-283.

[13] Keren, Gideon, and Peter Roelofsma. 1995. “Immediacy and Certainty in Intertemporal Choice.” *Organizational Behavior and Human Decision Making*, 63: 287-297.

[14] Laibson, David. 1997. “Golden Eggs and Hyperbolic Discounting.” *Quarterly Journal of Economics*, 112(2): 443-477.

[15] Machina, Mark. 1982. “Expected Utility Analysis Without the Independence Axiom.” *Econometrica*, 50, 277-323.

[16] O’Donoghue, Ted, and Matthew Rabin. 1999. “Doing It Now or Later.” *American Economic Review*, 89(1), 103-124.

[17] Dean, Mark,and Pietro Ortoleva. 2014. “Is it All Connected? A Testing Ground for Unified Theories of Behavioral Economics Phenomena,” working paper.

[18] Pennesi, Daniele. 2015. “Uncertain Discount and Hyperbolic Preferences,” Universite de Cergy-Pontoise, THEMA Working Paper, No. 2015–02.

[19] Saito, Kota. 2011. “Strotz Meets Allais: Diminishing Impatience and the Certainty Effect: Comment,” American Economic Review, 101 (5), 2271–75.

[20] Prelec, Darzen and George Loewenstein. 1991. “Decision Making Over Time and under Uncertainty: A Common Approach.” *Management Science*, 37(7): 770-786

[21] Sozou, Peter D. 1998. “On Hyperbolic Discounting and Uncertain Hazard Rates.” *Proceedings of the Royal Society London(Series B)*, 265(1409):2015-2020.

[22] Starmer, Chris. 2000. “Developments in Non-expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk.” *Journal of Economic Literature*, 38(2): 332-382.

## Appendix: Proofs of (ii), (iii), (ii’), and (iii’)

Step 1: (ii)

Substep 1.1: If $\succsim^r$ exhibits the certainty effect, then $\{\succsim_d\}$ is quasi-hyperbolic.
Proof of Substep 1.1: Choose any $x, y \in X$ and $t, s \in T$ such that $[x,t] \sim_t [y,s]$ and $t \leq s$. Then by definition, $(x,1) \sim^r (y, p(s|t))$. Fix $d$ such that $d < t$ to show $[x,t] \prec_d [y,s]$. Since $p$ is strictly decreasing, $p(t|d) < 1$. So the certainty effect implies that $(x, p(t|d)) \prec^r (y, p(s|t)p(t|d)) = (y, p(s|d))$. Then by definition, $[x,t] \prec_d [y,s]$.

Substep 1.2: If $\{\succsim_d\}$ exhibits quasi-hyperbolic discounting, then $\succsim^r$ exhibits the certainty effect.
Proof of Substep 1.2: Choose any $x, y \in X$ and $\mu \in [0,1]$ such that $(x,1) \sim^r (y,\mu)$. Fix $\eta \in (0,1)$ to show $(x,\eta) \prec^r (y,\eta\mu)$. Since $p$ is strictly decreasing and bijection to $[0,1]$, there exists $t$ such that $p(t) = \eta$. Also, there exists $s$ such that $s \geq t$ and $p(s) = \mu\eta$. Then, $p(s|t) = \mu$. Hence, $(x, p(0)) = (x,1) \sim^r (y,\mu) = (y, p(s|t))$, so that $[x,t] \sim_t [y,s]$, by definition. Therefore, if $\{\succsim_d\}$ is quasi-hyperbolic, then $[x,t] \prec_0 [y,s]$. So the definition shows that $(x,\eta) = (x, p(t)) \prec^r (y, p(s)) = (y, \eta\mu)$. □

Step 2: (iii)

Substep 2.1: If $\succsim^r$ satisfies the independence axiom, then $\{\succsim_d\}$ is temporally unbiased.
Proof of Substep 2.1: Choose any $x, y \in X$ and $t, s, d, d' \in T$ such that $[x,t] \succsim_d [y,s]$ to show $[x,t] \succsim_{d'} [y,s]$. Since $[x,t] \succsim_d [y,s]$, by definition $(x, p(t|d)) \succsim^r (y, p(s|d))$.

Case 1: Consider the case in which $d > d'$. By the independence axiom, $(x, p(t|d')) = (x, p(t|d)p(d|d')) \succsim^r (y, p(s|d)p(d|d')) = (y, p(s|d'))$. By the definition, $[x, t] \succsim_{d'} [y, s]$.
Case 2: Consider the case in which $d' > d$. Since $(x, p(t|d)) = (x, p(t|d')p(d'|d)) \succsim^r (y, p(s|d')p(d'|d)) = (y, p(s|d))$. Since $(x, p(t|d)) \succsim^r (y, p(s|d))$, by the independence axiom, $(x, p(t|d')) \succsim^r (y, p(s|d'))$. Hence, $[x, t] \succsim_{d'} [y, s]$.

Substep 2.2: If $\{\succsim_d\}$ is temporally unbiased, then $\succsim^r$ satisfies the independence axiom.
Proof of Substep 2.2: Choose any $x, y \in X$ and $\mu, \eta, \eta' \in [0, 1]$ such that $(x, \eta) \succsim^r (y, \eta\mu)$ to show $(x, \eta') \succsim^r (y, \eta'\mu)$.
Case 1: Consider the case in which $\eta' > \eta$. Since $p$ is strictly decreasing and bijection to $[0, 1]$, there exist $t, s, d \in T$ such that $s \geq t$, $p(t) = \eta$, $p(s) = \eta\mu$, and $p(d) = \eta/\eta'$. Then, $p(t|d) = \eta'$ and $p(s|d) = \eta'\mu$. Since $\{\succsim_d\}$ is temporally unbiased, $(x, \eta) \succsim^r (y, \eta\mu) \Leftrightarrow [x, t] \succsim_0 [y, s] \Leftrightarrow [x, t] \succsim_d [y, s] \Leftrightarrow (x, p(t|d)) \succsim^r (y, p(s|d)) \Leftrightarrow (x, \eta') \succsim^r (y, \eta'\mu)$.
Case 2: Consider the case in which $\eta' < \eta$. Since $p$ is strictly decreasing and bijection to $[0, 1]$, there exist $t, s, d \in T$ such that $s \geq t$, $p(t) = \eta'$, $p(s) = \eta'\mu$, and $p(d) = \eta'/\eta$. Then, $p(t|d) = \eta$ and $p(s|d) = \eta\mu$. Since $\{\succsim_d\}$ is temporally unbiased, $(x, \eta) \succsim^r (y, \eta\mu) \Leftrightarrow (x, p(t|d)) \succsim^r (y, p(s|d)) \Leftrightarrow [x, t] \succsim_d [y, s] \Leftrightarrow [x, t] \succsim_0 [y, s] \Leftrightarrow (x, p(t)) \succsim^r (y, p(s)) \Leftrightarrow (x, \eta') \succsim^r (y, \eta'\mu)$. □

Note that only Substeps 1.2, and 2.2 depend on the assumption that the set $T$ of time period is $\mathbb{R}_+$. When the set $T$ of time period is $\mathbb{Z}_+$, only Substeps 1.1 and 2.1 hold. So I obtain (ii') and (iii').