Consumption Theory: Problem Set 3

Duality, welfare measures, and taxation

Student version: questions only

Instructions. Answer both questions and justify your answers. Each question is worth 25 points; bracketed numbers give the points for each part. The total is 50 points.

1. **Duality and exact welfare measurement** **25 points**

   Let $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, $0 < \alpha < 1$, with fixed income $I > 0$. Initially $p^0 = (1, 1)$; first consider a price increase to $p^1 = (t, 1)$, $t > 1$. Write $u^0 = v(p^0, I)$ and $u^1 = v(p^1, I)$.

   $$CV = e(p^1, u^0) - I, \qquad EV = I - e(p^0, u^1), \qquad AV = \int_1^t x_1(q, 1, I)\, dq.$$

   (a) **[7]** Derive the expenditure function and both Hicksian demands for any $p_1, p_2 > 0$ and target utility $\bar{u} > 0$. Find $v(p, I)$ and verify $e(p, v(p, I)) = I$. Check that Roy's identity recovers Marshallian demand.

   (b) **[7]** Obtain explicit formulas for CV, EV, and AV in terms of (I, α, t). Explain the income-adjustment interpretation of each of CV and EV.

   (c) **[6]** Prove $CV > AV > EV > 0$ when $t > 1$. *Hint:* Set $z = \alpha \log t > 0$ and use $e^z > 1 + z$ and $e^{-z} > 1 - z$.

   (d) **[5]** Now let both prices rise from $(1, 1)$ to (r, s), where $r, s > 1$, while income remains fixed at I. Consider the Marshallian line integral

   $$AV_\gamma = \int_\gamma \big[x_1(p_1, p_2, I)\, dp_1 + x_2(p_1, p_2, I)\, dp_2\big].$$

   Calculate this integral separately along the following two paths:

   $$\gamma_{12}: \quad (1, 1) \longrightarrow (r, 1) \longrightarrow (r, s),$$
   $$\gamma_{21}: \quad (1, 1) \longrightarrow (1, s) \longrightarrow (r, s).$$

   Along γ_{12}, the price of good 1 increases first and then the price of good 2 increases. Along γ_{21}, the order is reversed. Each segment changes only the indicated price.

   Do the two calculations give the same answer? Determine whether the integral is independent of the path for every piecewise continuously differentiable path in the positive-price domain joining $(1, 1)$ to (r, s). Explain how your conclusion is consistent with both goods having positive income effects. Finally, compare the integral with CV and EV for this price change.

2. **Quasi-linear utility, welfare measures, and taxation** **25 points**

Consider a consumer with quasi-linear utility

$$u(x_1, x_2) = \phi(x_1) + x_2,$$

where ϕ is twice continuously differentiable on $(0, \infty)$, $\phi' > 0$, and $\phi'' < 0$. Income is $I > 0$, and prices are $(p_1, 1)$, so good 2 is the numeraire. Consumption of both goods must be nonnegative. *Assume that all Marshallian and Hicksian choices considered below are interior, including the compensated choices along each price change.*

(a) **[4]** Derive $x_1(p_1, I)$ and $x_2(p_1, I)$. Show that demand for good 1 does not depend on income, and hence write it as $x_1(p_1)$.

(b) **[5]** Derive the indirect utility function $v(p_1, I)$ and the expenditure function $e(p_1, \bar{u})$.

(c) **[4]** Show that $\bar{x}_1(p_1, \bar{u}) = x_1(p_1)$ for every target utility satisfying the interiority assumption. Interpret this equality using the Slutsky equation.

(d) **[6]** Consider a price change from p_1^0 to p_1^1, keeping income fixed at I, and let $u^j = v(p_1^j, I)$ for $j = 0, 1$. Use

$$CV = e(p_1^1, u^0) - I, \qquad EV = I - e(p_1^0, u^1), \qquad AV = \int_{p_1^0}^{p_1^1} x_1(t)\, dt.$$

Prove that $CV = AV = EV$. These measures are positive for a price increase; use the same signed definitions if the price falls.

(e) **[6]** Now let $\phi(x_1) = 2\sqrt{x_1}$ and $I = 20$, and suppose the price increases from $p_1^0 = 1$ to $p_1^1 = 4$. Compute CV, EV, and AV.