

Consumption Theory: Problem Set 2

Demand restrictions, elasticities, and welfare

Student version: questions only

Instructions. Answer all four questions and justify your answers. Each question is worth 25 points; bracketed numbers give the points for each part. The total is 100 points.

Reading. Saito, *Lecture Note on Consumption Theory* (September 15, 2026), Sections 7–14 and the definition of the line integral in Appendix B. Use x for ordinary demand and x^h for compensated demand. All differentiation holds other arguments fixed unless stated otherwise.

Compensated substitution and asymmetric ordinary demand

25 points

Write $x(p, I)$ for Marshallian demand and $\bar{x}(p, \bar{u})$ for Hicksian (compensated) demand, as in the handout. At the point under consideration set $\bar{u} = v(p, I)$ and

$$s_{ij} = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} = \frac{\partial x_i(p, I)}{\partial p_j} + \frac{\partial x_i(p, I)}{\partial I} x_j(p, I).$$

Assume the required derivatives exist and the expenditure function is twice continuously differentiable wherever symmetry is used.

- (a) [6] Prove Hicksian symmetry, $s_{ij} = s_{ji}$, and derive

$$\frac{\partial x_i}{\partial p_j} - \frac{\partial x_j}{\partial p_i} = x_i \frac{\partial x_j}{\partial I} - x_j \frac{\partial x_i}{\partial I}.$$

Explain why symmetry of compensated demand need not imply symmetry of ordinary demand, or even equality of the signs of its two cross-price effects.

- (b) [9] Consider $u(x_1, x_2) = (x_1 - 1)(x_2 + 1)$ on $X = \{(x_1, x_2) : x_1 \geq 1, x_2 \geq 0\}$. Consider prices and income around $p_1 = p_2 = 1, I = 4$, where demand is interior relative. Derive the Marshallian demand, and calculate at this point $x, \frac{\partial x_1}{\partial I}, \frac{\partial x_2}{\partial I}, \frac{\partial x_1}{\partial p_2}, \frac{\partial x_2}{\partial p_1}, s_{12}$, and s_{21} . Verify that this is utility-maximizing demand, not merely a proposed formula.
- (c) [5] Calculate both income elasticities at the price–income pair in (b). Identify which good is a luxury and which is a necessity. Recall that the income elasticity is $\eta_i = (\partial x_i / \partial I)(I/x_i)$; a luxury satisfies $\eta_i > 1$, whereas a necessity satisfies $0 < \eta_i < 1$.

Next, evaluate the following claim:

“Opposite-signed cross-price effects of Marshallian demand require at least one good to be inferior.”

Here, *opposite-signed cross-price effects* means that, at the same initial price–income pair,

$$\frac{\partial x_1}{\partial p_2} > 0 \quad \text{and} \quad \frac{\partial x_2}{\partial p_1} < 0,$$

or the reverse sign pattern. This means raising the price of good 2 increases demand for good 1, whereas raising the price of good 1 decreases demand for good 2. These are two separate price-change experiments.

The claim asserts that such a pattern is possible only if $\partial x_1 / \partial I < 0$ or $\partial x_2 / \partial I < 0$. Is this true or false? Use the example in (b) and the Slutsky equations to justify your answer, explicitly comparing the substitution and income effects in the two experiments.

- (d) [5] In a two-good economy, use homogeneity of Hicksian demand and $s_{11} \leq 0$ to show $s_{12} \geq 0$. Must the inequality be strict? Give a utility function and its Hicksian demands for which $s_{12} = 0$.

What can observed demand reject?**25 points**

Normalize the price of good 2 to one, and write $p = p_1/p_2$ and $m = I/p_2$. An economist proposes the exact local demand equation

$$x_1(p, m) = -ap + bm, \quad a = 400, \quad b = 0.62.$$

The demand for good 2 is determined by the budget constraint: $x_2(p, m) = m - px_1(p, m)$. Assume that the formulas hold on an open neighborhood of each observation considered, with nonnegative demands.

- (a) [6] Calculate the own Slutsky term s_{11} in normalized coordinates. What upper bound on x_1 is necessary for $s_{11} \leq 0$?
- (b) [6] Evaluate demand, budget feasibility, and s_{11} at $(p, m) = (1, 1000)$ and $(1, 2000)$. Can utility maximization be rejected at either observation?
- (c) [7] Write the corresponding demand system at arbitrary positive (p_1, p_2, I) . Verify homogeneity and Walras' law. At $p_2 = 1$, show that

$$S = s_{11} \begin{pmatrix} 1 & -p \\ -p & p^2 \end{pmatrix}, \quad z^\top S z = s_{11}(z_1 - pz_2)^2.$$

Hint: Homogeneity and Walras' law together give $Sp = 0$ and $p^\top S = 0$; in these vector identities use the price vector $(p, 1)^\top$.

- (d) [6] For comparison, a three-good demand system has the following estimated Slutsky matrix at prices $(1, 1, 1)$:

$$S_3 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 2 \\ 2 & 0 & -2 \end{pmatrix}.$$

Check its left and right price-null restrictions, and prove $z^\top S_3 z \leq 0$ for every $z \in \mathbb{R}^3$. Is the matrix consistent with smooth utility-maximizing demand? Does passing the quadratic-form test alone establish WARP for an entire demand system?