

Lecture Note on Consumption Theory

Kota SAITO

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A: Theoretical Foundation of Rational Choice

1 Rational Choice

We model a consumer's behaviour by specifying a preference relation on the set of alternatives. Let X denote the set of all possible alternatives (the choice set). In consumer theory, X is typically a subset of $\mathbb{R}_+^N$, representing bundles of N goods.

Definition 1.1 (Preference relation). A preference relation $\succsim$ is a binary relation on X . We read $x \succsim y$ as “the agent weakly prefers x to y .” Two derived relations are defined as follows:

- Strict preference: $x \succ y$ if $x \succsim y$ and not $y \succsim x$.
- Indifference: $x \sim y$ if $x \succsim y$ and $y \succsim x$.

In economics, we elicit $\succsim$ through incentivised choices rather than self-reported surveys, since incentivised choices reveal preferences more reliably. An agent chooses x from $S \subset X$ if x is the $\succsim$ -best element of S .

Definition 1.2 (Rational preference relation (MWG Def. 1.B.1)). A preference relation $\succsim$ on X is rational if it satisfies:

- (i) **Completeness**: For all $x, y \in X$, $x \succsim y$ or $y \succsim x$ (or both).
- (ii) **Transitivity**: For all $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$.

Interpretation of rationality

When X is finite, completeness and transitivity together guarantee that the agent can rank all alternatives from best to worst. Completeness rules out “I cannot compare these”; transitivity rules out preference cycles.

Remark 1. *Completeness of $\succsim$ implies asymmetry of $\succ$ (i.e., there exist no x, y such that $x \succ y$ and $y \succ x$).*

Remark 2 (Limits of transitivity). *The transitivity of $\succsim$ implies transitivity of $\sim$. This can be unreasonable: let x_i be a cup of coffee with i grams of sugar. A consumer cannot detect a 1-gram difference, so $x_i \sim x_{i+1}$ for every i . Transitivity would then imply $x_1 \sim x_{100}$ —but 100 extra grams is clearly detectable. Rationality is therefore a modelling idealisation rather than a literal description of cognition.*

1.1 Choice Structures and Choice Rules (MWG §1.B–1.C)

So far we have described a consumer's preference directly via the preference relation $\succsim$. An alternative—and complementary—approach is to model behaviour through a choice rule, asking: given a set of feasible alternatives, what does the agent actually choose?

Definition 1.3 (Choice structure). A choice structure is a pair $(\mathcal{B}, C(\cdot))$ where:

- (i) $\mathcal{B}$ is a collection of nonempty subsets of X (the possible choice sets or budget sets);
- (ii) $C(\cdot)$ is a choice rule: a mapping that assigns to each $B \in \mathcal{B}$ a nonempty set $C(B) \subseteq B$ (the chosen elements from B).

Interpretation

$C(B)$ is the set of alternatives the agent might select when the feasible set is B . Allowing $C(B)$ to be a set (rather than a single element) captures indifference: if x and y are both in $C(B)$, the agent is willing to choose either.

Example 1 (Walrasian budget sets). In consumer theory, $X = \mathbb{R}_+^L$. For prices $p \in \mathbb{R}_{++}^L$ and wealth $w > 0$, define the Walrasian budget set

$$B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}.$$

The family $\mathcal{B} = \{B_{p,w} : p \in \mathbb{R}_{++}^L, w > 0\}$ is the standard collection of budget sets in the consumer's problem. The Walrasian demand correspondence $x(\cdot, \cdot)$ defines a choice rule via $C(B_{p,w}) = x(p, w)$.

The Weak Axiom of Revealed Preference for Choice Structures

Definition 1.4 (Weak Axiom of Revealed Preference — abstract form (MWG Def. 1.C.1)). The choice structure $(\mathcal{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (WARP) if the following holds: for any $B, B' \in \mathcal{B}$ and any $x, y \in B \cap B'$,

$$x \in C(B) \text{ and } y \in C(B') \implies x \in C(B').$$

Note that the requirements depend on the sets $\mathcal{B}$: As $\mathcal{B}$ becomes richer, the requirements become stronger.

Intuition for WARP

If the agent chooses x from B (when y was also available), then x is revealed preferred to y . WARP says: whenever y is later chosen from some B' that also contains x , the agent must be willing to choose x from B' as well. In other words, the agent cannot simultaneously reveal $x \succ y$ and $y \succ x$.

The Relationship Between Preferences and Choice Rules (MWG §1.C)

Definition 1.5 (Rationalization). A choice structure $(\mathcal{B}, C(\cdot))$ is rationalized by a preference relation $\succsim$ if

$$C(B) = C_{\succsim}(B) \equiv \{x \in B : x \succsim y \text{ for all } y \in B\} \quad \text{for every } B \in \mathcal{B}.$$

That is, the chosen set equals the set of $\succsim$ -maximal elements of B .

Proposition 1 (Rationality implies WARP (MWG Prop. 1.C.1)). *If $\succsim$ is a rational preference relation and $(\mathcal{B}, C_{\succsim}(\cdot))$ is the induced choice structure, then $(\mathcal{B}, C_{\succsim}(\cdot))$ satisfies WARP.*

Proof. Let $B, B' \in \mathcal{B}$ and $x, y \in B \cap B'$. Suppose $x \in C_{\succsim}(B)$ and $y \in C_{\succsim}(B')$.

- $x \in C_{\succsim}(B)$ implies $x \succsim z$ for all $z \in B$; in particular, $x \succsim y$ (since $y \in B$).
- $y \in C_{\succsim}(B')$ implies $y \succsim z$ for all $z \in B'$; in particular, $y \succsim x$ (since $x \in B'$).
- By completeness and transitivity, $x \succsim y$ and $y \succsim x$ give $x \sim y$. Since $y \succsim z$ for all $z \in B'$ and $x \sim y$, we have $x \succsim z$ for all $z \in B'$ (by transitivity: $x \sim y \succsim z$ implies $x \succsim z$).
- Therefore $x \in C_{\succsim}(B')$. □

Proposition 2 (WARP implies rationalizability (MWG Prop. 1.C.2)). *Suppose $\mathcal{B}$ contains all nonempty subsets of X of size at most three. If $(\mathcal{B}, C(\cdot))$ satisfies WARP, then there exists a unique rational preference relation $\succsim$ that rationalizes $(\mathcal{B}, C(\cdot))$.*

Proof. Define a relation $\succeq$ on X by

$$x \succeq y \iff x \in C(\{x, y\}).$$

Since choices are nonempty, this relation is reflexive and complete.

Step 1: The relation rationalizes the choice rule. We show that, for every $B \in \mathcal{B}$,

$$C(B) = \{x \in B : x \succeq y \text{ for every } y \in B\}.$$

First, let $x \in C(B)$ and take any $y \in B$. Choose $z \in C(\{x, y\})$. Since x and z belong to both B and $\{x, y\}$, WARP implies $x \in C(\{x, y\})$. Thus $x \succeq y$.

Conversely, suppose $x \in B$ and $x \succeq y$ for every $y \in B$. Choose $z \in C(B)$, which exists because choices are nonempty. By the definition of $\succeq$, we have $x \in C(\{x, z\})$. Applying WARP to $\{x, z\}$ and B gives $x \in C(B)$.

Step 2: The relation is transitive. Suppose $x \succeq y$ and $y \succeq z$. The set $T = \{x, y, z\}$ belongs to $\mathcal{B}$, so $C(T)$ is nonempty. We claim that $x \in C(T)$.

If $y \in C(T)$, then $x \in C(\{x, y\})$ and WARP imply $x \in C(T)$. If $z \in C(T)$, then $y \in C(\{y, z\})$ and WARP first imply $y \in C(T)$, and the preceding argument then implies $x \in C(T)$. Since at least one of x , y , and z belongs to $C(T)$, the claim follows.

By Step 1, $x \in C(T)$ implies $x \succeq z$. Hence $\succeq$ is transitive and therefore rational.

Step 3: Uniqueness. Let $\succeq'$ be any rational preference relation that rationalizes the choice rule. Because it rationalizes choices from every binary set,

$$x \succeq' y \iff x \in C(\{x, y\}) \iff x \succeq y.$$

Thus $\succeq' = \succeq$. □

Notice that three-element sets are used only in Step 2 to establish transitivity. Binary sets alone imply uniqueness, conditional on the existence of a rational rationalization. Without the three element sets, WARP does not allow us to conclude the transitivity.

Summary: Preference relations vs. choice rules

- Rational preferences always generate a choice rule satisfying WARP (Proposition above).
- When $\mathcal{B}$ is sufficiently rich (contains all pairs and triples), WARP is sufficient to recover a unique rational preference relation.
- For Walrasian budget sets, however, $\mathcal{B}$ is not rich enough: WARP is necessary but generally not sufficient for rationalizability when $L \geq 3$ goods are present. (See Section 11 for the demand-theoretic version of WARP.)

2 Utility Function

A preference relation $\succsim$ is powerful but abstract. To work with it analytically, we seek a real-valued function that encodes the preference ordering.

Definition 2.1 (Utility function). A function $u : X \rightarrow \mathbb{R}$ represents $\succsim$ if

$$x \succsim y \iff u(x) \geq u(y) \quad \text{for all } x, y \in X.$$

Remark 3 (Ordinal utility). *If u represents $\succsim$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing, then $\phi \circ u$ also represents $\succsim$. Only the ordering of utility values matters, not their magnitude. This is the foundation of ordinal utility theory: utility numbers are a convenient representation, not a cardinal measure of wellbeing.*

Proposition When X is finite, $\succsim$ can be represented by a utility function if and only if $\succsim$ is rational.

Proof. For each $x \in X$, define $W(x)$ as the set of elements in X that x is weakly preferred to. That is,

$$W(x) = \{y \in X \mid x \succsim y\}.$$

Then define $u(x) = |W(x)| = \sum_{y \in W(x)} 1$. □

When X is infinite, even if it is countable, the previous proof does not work because $u(x)$ can be infinity and it can be that $u(x) = +\infty = u(y)$ for $x \succ y$.

Proposition When X is countable, the results also holds: $\succsim$ can be represented by a utility function if and only if $\succsim$ is rational.

Proof. Consider how to construct a utility function. Enumerate X by $\{x_1, x_2, x_3, \dots\}$. Define $u(x) = \sum_{x_i \in W(x)} (\frac{1}{2})^i$. (Now the value is bounded.) □

2.1 Lexicographic preference

When X is uncountable, the result may not be the case. Even if $\succsim$ is rational, it may not have a utility representation.

Example 2 (Lexicographic preference on $\mathbb{R}^2$). For any pairs in $\mathbb{R}^2$, define

$$(x_1, x_2) \succ (y_1, y_2) \quad \text{if } x_1 > y_1, \text{ or } (x_1 = y_1 \text{ and } x_2 > y_2).$$

Let $x \succsim y$ if not $(y_1, y_2) \succ (x_1, x_2)$.

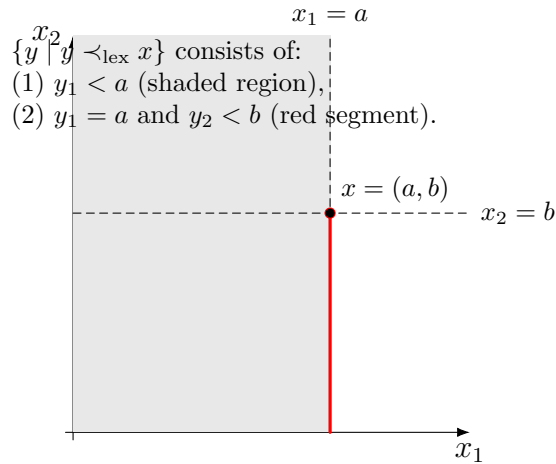
Proposition Lexicographic preference relations are rational. However, for any Lexicographic preference $\succ$ relation, there is no $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $x \succ y \implies u(x) > u(y)$.

Proof. It is easy to show that $\succsim$ is rational.

Proof by contradiction. For any $x \in \mathbb{R}$ such that $(x, 1) \succ (x, 0)$. Then $u(x, 0) < u(x, 1)$. Choose a rational number $r \in (u(x, 0), u(x, 1))$. Denote such r by r_x . For any $x, y \in \mathbb{R}$ such that $x < y$, I have $u(x, 0) < r_x < u(x, 1) < u(y, 0) < r_y < u(y, 1)$. So $r_x < r_y$. So for each $\mathbb{R}$, I obtain one to one mapping to $\mathbb{Q}$. This is a contradiction. $\square$

Remark 4 (A countability condition for utility representation). When X is uncountable, completeness and transitivity alone do not guarantee the existence of a utility representation. A standard additional condition is order separability: there exists a countable subset $Y \subseteq X$ such that whenever $x \succ y$, there is some $z \in Y$ satisfying $x \succeq z \succeq y$.

Intuitively, the countable set Y is rich enough to “separate” every strictly ordered pair: whenever x is strictly preferred to y , at least one reference alternative from Y lies between them in the preference ordering. This allows an uncountable preference ordering to be encoded using only countably many reference points, and hence by real numbers. Lexicographic preferences fail this type of countability condition, which is why they need not admit a real-valued utility representation.



2.2 Continuous utility function

For uncountable X (e.g. $X = \mathbb{R}_+^N$), an additional regularity condition is needed.

Definition 2.2 (Continuous preference relation). $\succsim$ is continuous if for every $y \in X$, the upper contour set $\{x \in X : x \succsim y\}$ and the lower contour set $\{x \in X : y \succsim x\}$ are both closed. (This requires a topology on X ; the standard choice is the Euclidean topology.)

Remark 5 (Intuition for continuity). *Suppose that $\succeq$ is complete. Then continuity means that strict preference is robust to sufficiently small changes. If $y \succ x$, then completeness implies that $x \not\succeq y$. Hence the set $\{x \in X : y \succ x\}$ is the complement of the closed set $\{x \in X : x \succeq y\}$, and is therefore open. Thus, if $y \succ x$, every alternative sufficiently close to x is also strictly worse than y .*

For example, let $x = (p, q)$, where p is price and q is quality. If $y \succ x$, continuity means that sufficiently small changes in the price or quality of x do not reverse the strict preference. That is, there exists some $\varepsilon > 0$ such that $\|x' - x\| < \varepsilon$ implies $y \succ x'$. Thus continuity rules out abrupt reversals of strict preference caused by arbitrarily small changes in an alternative.

Definition 2.3 (Separable topological space). A topological space X is separable if there exists a countable subset $Y \subseteq X$ whose closure equals X .

Definition 2.4 (Connected topological space). A topological space X is connected if there exists no disjoint, nonempty, closed sets A and B such that $A \cup B = X$.¹

The following is a general sufficient condition for the existence of a continuous utility representation.

Theorem 1 (Eilenberg-Debreu). *Let X be a connected and separable topological space. A preference relation $\succeq$ on X is rational and continuous if and only if there exists a continuous utility function $u : X \rightarrow \mathbb{R}$ that represents $\succeq$.*²

For consumer theory, we usually work with subsets of Euclidean space. In this case we can use the following simpler version.

Theorem 2 (Continuous utility representation in Euclidean space). *Let $X \subseteq \mathbb{R}^N$. If $\succeq$ is rational and continuous, then there exists a continuous utility function $u : X \rightarrow \mathbb{R}$ that represents $\succeq$.*

2.3 Monotonic and continuous utility function

- Let $X = \mathbb{R}_+^N$. (Then X is connected.)
- A preference relation $\succsim$ is monotonic if for any $x, y \in X$, $x_i \geq y_i$ for all i implies $x \succsim y$; $x_i \geq y_i$ for all i and $x_j > y_j$ for some j implies $x \succ y$.

¹The definition can be stated under the assumption that both A and B are open sets, instead of closed sets.

²Debreu's version assumes a second countable topological space, instead of connected and separable topological space

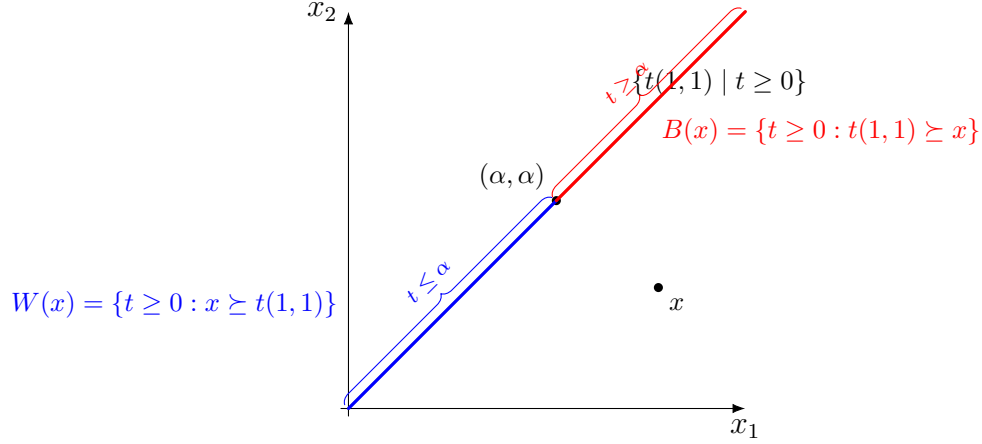


Figure 1: Add a indifference curve that goes through x and (α, α)

- A utility function u is monotonic if $x_i \geq y_i$ for all i implies $u(x) \geq u(y)$; $x_i \geq y_i$ for all i and $x_j > y_j$ for some j implies $u(x) > u(y)$.

Proposition Let $X = \mathbb{R}_+^N$. A preference relation $\succsim$ is complete, transitive, continuous, and monotonic if and only if there exists a monotonic continuous utility function u that represents $\succsim$.

Proof. Fix $x \in X$. Find a number α such that $x \sim (\alpha, \dots, \alpha)$. Define $u(x) = \alpha$.

The purpose of the following proof is to show the existence of such α and its uniqueness. Define $B(x) = \{t \in \mathbb{R}_+ | t(1, \dots, 1) \succsim x\}$ and $W(x) = \{t \in \mathbb{R}_+ | x \succsim t(1, \dots, 1)\}$.

- Show that both $B(x)$ and $W(x)$ are nonempty and closed.
- Show also that $B(x) \cup W(x) = \mathbb{R}_+$.
- Use the fact that $\mathbb{R}_+$ is connected to show $B(x) \cap W(x) \neq \emptyset$. Let α be an element of $B(x) \cap W(x)$.
- By monotonicity, we can show that α is unique and u is monotonic
- By using transitivity, we can show that u represents $\succsim$.
- By using the continuity of $\succsim$, we can show that u is continuous.

□

2.4 Differentiable utility function

Let $X = \mathbb{R}_{++}^N$, and suppose that $\succeq$ is complete, transitive, continuous, and strictly monotone. Fix an integer $r \geq 1$. A function is C^r if its derivatives through order r exist and are continuous. The relevant question is whether $\succeq$ admits at least one C^r utility representation,

not whether every representation is C^r : a strictly increasing transformation can destroy differentiability without changing preferences.

A smoothness condition on preferences. For every indifferent pair $(\bar{x}, \bar{y})$, suppose there exist an open neighborhood $W \subseteq X \times X$ of $(\bar{x}, \bar{y})$ and a function $F \in C^r(W)$ such that

$$x \sim y \iff F(x, y) = 0 \quad \text{for all } (x, y) \in W,$$

with $\nabla_{(x,y)} F(\bar{x}, \bar{y}) \neq 0$. Thus, indifference is locally described by one smooth equation whose gradient does not vanish. This is what it means for $\mathcal{I} = \{(x, y) \in X \times X : x \sim y\}$ to be a C^r hypersurface. The gradient is taken with respect to all $2N$ coordinates of (x, y) .

Theorem (smooth utility representation). Under the maintained assumptions, the smoothness condition above holds if and only if there exists a utility representation $u \in C^r(X)$ satisfying

$$\nabla u(x) \neq 0 \quad \text{for every } x \in X.$$

The classical reference is Debreu (1972), Smooth Preferences; the formulation above follows Mas-Colell and McLennan (1995).

A simpler sufficient condition: compensating quantities. With two goods, fix a reference bundle $x = (x_1, x_2)$ and a quantity y_1 of good 1. Let $c(x_1, x_2, y_1)$ be the quantity of good 2 needed to maintain indifference:

$$(x_1, x_2) \sim (y_1, y_2) \iff y_2 = c(x_1, x_2, y_1).$$

A sufficient condition is as follows: For every $\bar{x} \sim \bar{y}$, there exist an open neighborhood

$$U \subseteq \mathbb{R}_{++}^3$$

of $(\bar{x}_1, \bar{x}_2, \bar{y}_1)$ and a function

$$c : U \rightarrow \mathbb{R}_{++}$$

that is C^r , such that

$$(x_1, x_2) \sim (y_1, c(x_1, x_2, y_1)) \quad \text{for every } (x_1, x_2, y_1) \in U.$$

The defining equation is then $F(x, y) = y_2 - c(x_1, x_2, y_1)$, with $\partial F / \partial y_2 = 1$. For perfect substitutes, for example, $c(x_1, x_2, y_1) = x_1 + x_2 - y_1$ wherever this quantity is positive.

Joint smoothness matters: varying y_1 traces one indifference curve, whereas varying (x_1, x_2) changes the reference bundle. Requiring a smooth solution for y_2 everywhere is stronger than necessary; the general condition allows the coordinate solved for to vary across neighborhoods.

3 Indifference Curves

Consider an economy with N goods. The consumption set $X \subseteq \mathbb{R}^N$ is the set of feasible commodity bundles. Throughout, we fix a utility function $u : X \rightarrow \mathbb{R}$ representing the consumer's preferences.

Definition 3.1 (Indifference curve and upper contour set). For a utility level $a \in \mathbb{R}$, the indifference curve at level a is

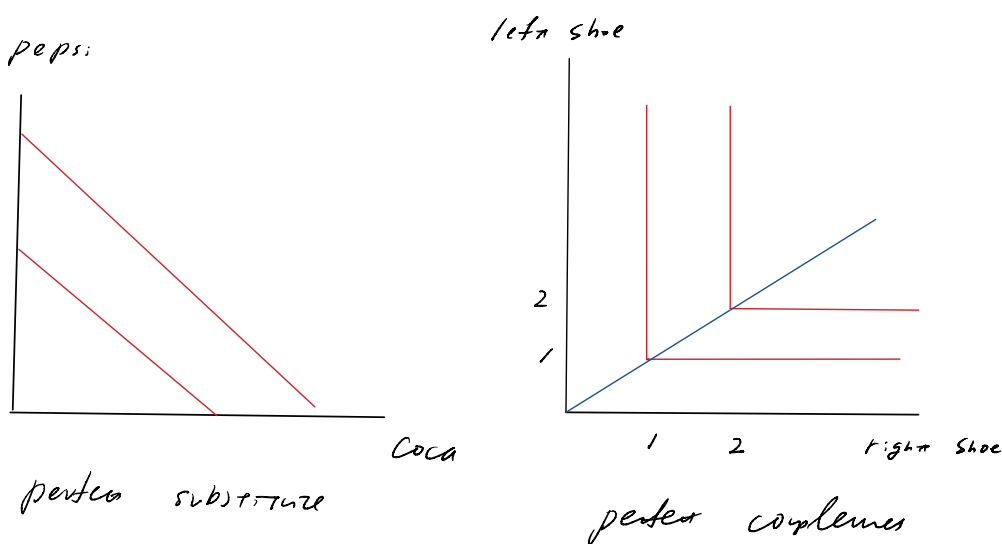
$$I(a) = \{x \in X : u(x) = a\},$$

and the upper contour set is $U(a) = \{x \in X : u(x) \geq a\}$.

Two polar cases illustrate the range of possible preference structures.

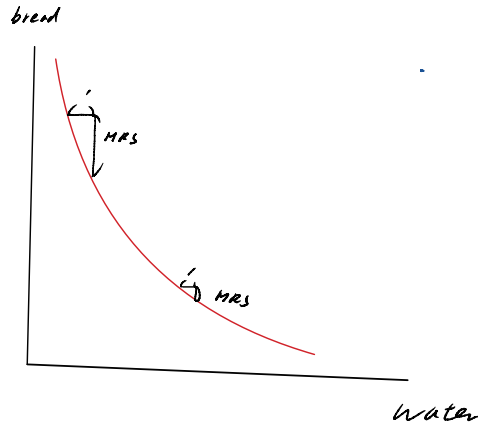
Example 3 (Perfect substitutes). *Coke and Pepsi for a consumer who is indifferent between them: $u(x_1, x_2) = x_1 + x_2$. Indifference curves are straight lines with slope -1 .*

Example 4 (Perfect complements). *Left and right shoes: $u(x_1, x_2) = \min\{x_1, x_2\}$. Indifference curves are L-shaped.*

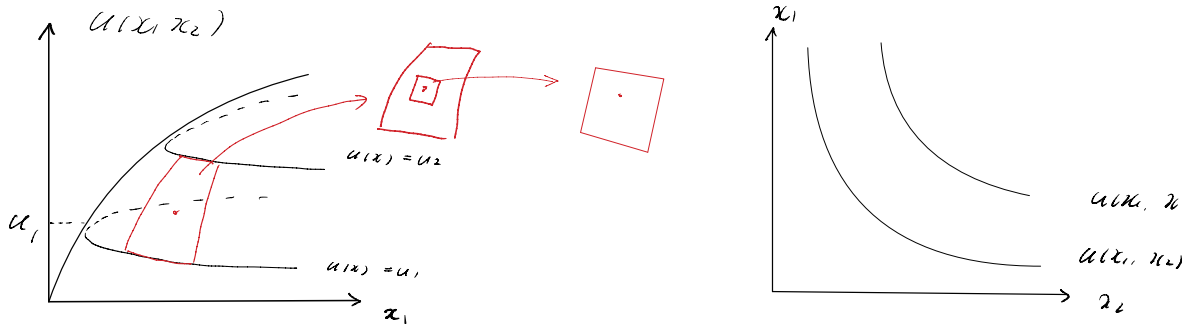


In most economic applications, indifference curves lie between these two extremes and exhibit a diminishing marginal rate of substitution.

Definition 3.2 (Marginal Rate of Substitution: intuitive definition). The marginal rate of substitution of good 1 for good 2 at a bundle x , denoted $MRS_{12}(x)$, is the rate at which the consumer is willing to give up good 2 per unit of additional good 1 while remaining on the same indifference curve. (We provide a formal definition later.)



4 Marginal Utility and MRS



Let $X \subseteq \mathbb{R}^N$ be the consumption set and let $u : X \rightarrow \mathbb{R}$ be a differentiable utility function. Write $x = (x_1, \dots, x_N)$.

Definition 4.1 (Marginal utility). For each good $i \in \{1, \dots, N\}$, the marginal utility of good i at bundle x is

$$MU_i(x) := \frac{\partial u(x)}{\partial x_i}.$$

MU_i : marginal increase of utility when the consumption of good i increases marginally.

Remark 6 (First-order approximation / local linearization). If u is differentiable at x and the bundle changes by a small vector $\Delta x \in \mathbb{R}^N$, then

$$\Delta u \equiv u(x + \Delta x) - u(x) \approx \sum_{i=1}^N \frac{\partial u(x)}{\partial x_i} \Delta x_i = \nabla u(x) \cdot \Delta x.$$

This is the multivariate analogue of the tangent-line approximation in one variable.

Intuitive Explanation. If the consumption of good i alone increases by Δx_i , utility changes by approximately $\Delta u \approx \frac{\partial u}{\partial x_i} \Delta x_i$. When several goods change simultaneously (good i increases by Δx_i), the linear case $u(x) = \sum_i a_i x_i$ gives $\Delta u = \sum_i a_i \Delta x_i$ exactly. For a smooth u , the function looks approximately linear locally, so the same formula holds to first order:

Total differential approximation

$$\Delta u \approx \sum_{i=1}^N \frac{\partial u}{\partial x_i} \Delta x_i.$$

Taking derivatives thus lets us analyse a local linear model of any smooth utility function.

Proposition 3 (Total differential). Let $X \subseteq \mathbb{R}^N$ be open and let $u : X \rightarrow \mathbb{R}$ be differentiable. Consider any differentiable path

$$x(t) = (x_1(t), \dots, x_N(t)) \in X.$$

Then the composite function $t \mapsto u(x(t))$ is differentiable and

$$\frac{d}{dt} u(x(t)) = \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x(t)) \frac{dx_i(t)}{dt}.$$

Equivalently, in differential notation,

$$du = \sum_{i=1}^N \frac{\partial u}{\partial x_i} dx_i.$$

4.1 MRS and marginal utility

Let $N = 2$. Along a change (dx_1, dx_2) , the total differential is

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2.$$

Assuming $du = 0$ (i.e., the same indifference curve), which is nonzero by assumption,

$$MRS_{12}(v) = -\frac{dx_2}{dx_1} = \frac{\partial u / \partial x_1}{\partial u / \partial x_2}.$$

Definition 4.2 (Marginal rate of substitution). Fix a utility level $\bar{u}$ and consider the indifference curve $\{x \in X : u(x) = \bar{u}\}$. At a differentiable point x on this curve with $MU_2(x) \neq 0$, the marginal rate of substitution of good 1 for good 2 is

$$MRS_{12}(x) := -\frac{dx_2}{dx_1} \Big|_{du=0} = \frac{MU_1(x)}{MU_2(x)}.$$

Proposition 4 (MRS is ordinal). *Let $v = \phi \circ u$ where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and strictly increasing. Then at any x with $MU_2(x) \neq 0$,*

$$MRS_{12}^v(x) = MRS_{12}^u(x).$$

Proof. By the chain rule, $\frac{\partial v}{\partial x_i} = \phi'(u(x)) \frac{\partial u}{\partial x_i}$ for $i = 1, 2$. Therefore

$$\frac{\partial v / \partial x_1}{\partial v / \partial x_2} = \frac{\phi'(u(x)) \partial u / \partial x_1}{\phi'(u(x)) \partial u / \partial x_2} = \frac{\partial u / \partial x_1}{\partial u / \partial x_2} \equiv \frac{MU_1(x)}{MU_2(x)}.$$

□

Definition 4.3 (Law of diminishing MRS). The law of diminishing marginal rate of substitution states that MRS_{12} decreases as x_1 increases along an indifference curve. Geometrically, this is equivalent to the indifference curves being convex (bowing toward the origin).

Remark 7 (Why convexity is a reasonable assumption). *If indifference curves are not convex, demand may respond discontinuously to price changes. Empirically, observed demand tends to move continuously with prices, which is consistent with convexity.*

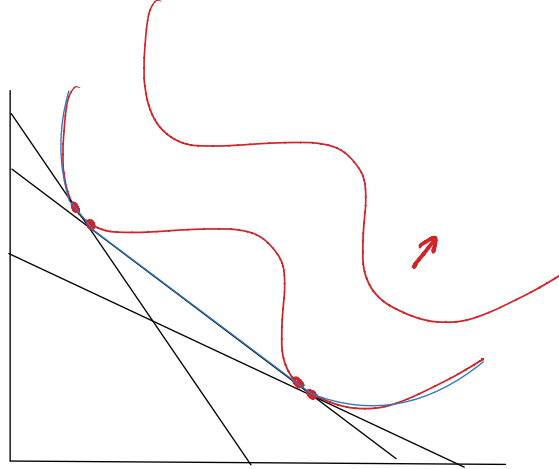


Figure 2: As price changes (i.e., as the budget line becomes less steep), the chosen point changes from the left upper points to right lower points drastically.

Moreover, even when true indifference curves are non-convex, a convexification (taking the convex hull of upper contour sets) leaves the Walrasian demand function nearly unchanged, so convexity is approximately without loss of generality.

4.2 Convex preferences and the law of decreasing MRS

Given x , the law of decreasing MRS is geometrically equivalent to the convexity of the set of $\{y \in X | y \succsim x\}$. This can be formalized as follows.

Definition 4.4. $\succsim$ is *convex* if $\{y \in X | y \succsim x\}$ is convex for any $x \in X$.³

4.3 Convexity of $\succ$ if and only if quasi-concave utility

Assume for simplicity that X is a convex set.

Definition 4.5. A function f is *quasi-concave* if for any $x, y \in X$ and $\alpha \in [0, 1]$ such that

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}.$$

A function f is *concave* if for any $x, y \in X$ and $\alpha \in [0, 1]$ such that

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y).$$

Example 5 (Quasi-concave but not concave). Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = x^3.$$

Since f is strictly increasing, it is quasi-concave. Indeed, for any $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, the point $\alpha x + (1 - \alpha)y$ lies between x and y . Hence

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}.$$

To see this take any $x, y \in \mathbb{R}$ and any $\alpha \in [0, 1]$. Since $\alpha x + (1 - \alpha)y$ lies between x and y ,

$$\alpha x + (1 - \alpha)y \geq \min\{x, y\}.$$

Because $t \mapsto t^3$ is strictly increasing,

$$(\alpha x + (1 - \alpha)y)^3 \geq (\min\{x, y\})^3 = \min\{x^3, y^3\}.$$

Therefore,

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\},$$

so $f(x) = x^3$ is quasi-concave.

However, f is not concave. For example, taking $x = 0$, $y = 2$, and $\alpha = 1/2$,

$$f(1) = 1 < 4 = \frac{1}{2}f(0) + \frac{1}{2}f(2).$$

Thus quasi-concavity is weaker than concavity.

For each $c \in \mathbb{R}$, define the upper level set (or upper contour set) of f by

$$U_f(c) = \{x \in X : f(x) \geq c\}.$$

Proposition A function $f : X \rightarrow \mathbb{R}$ is quasi-concave if and only if $U_f(c)$ is convex for every $c \in \mathbb{R}$.

³Alternatively, we can define convexity as $\{y \in X | y \succ x\}$ is convex for any x , assuming that $\succsim$ is complete and transitive and $x \succ y$ if and only if not $y \succsim x$.

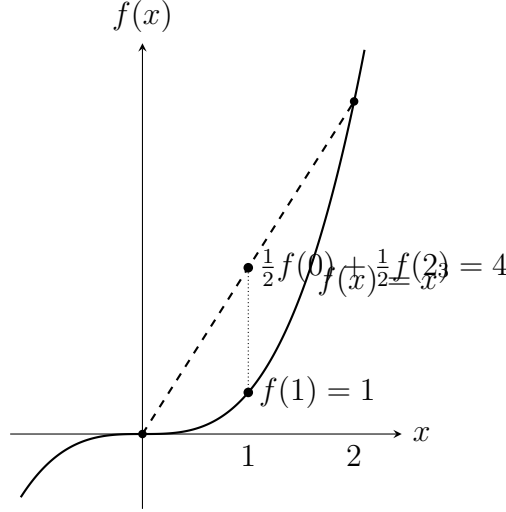


Figure 3: $f(x) = x^3$ is quasi-concave but not concave. At the midpoint of 0 and 2, the graph lies strictly below the chord.

Proof.

Step 1: Suppose that f is quasi-concave. We show that $U_f(c)$ is convex for every $c \in \mathbb{R}$.

Fix $c \in \mathbb{R}$ and take $x, y \in U_f(c)$. Then

$$f(x) \geq c \quad \text{and} \quad f(y) \geq c.$$

Hence, for every $\alpha \in [0, 1]$, quasi-concavity implies

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\} \geq c.$$

Therefore,

$$\alpha x + (1 - \alpha)y \in U_f(c),$$

so $U_f(c)$ is convex.

Step 2: Suppose that $U_f(c)$ is convex for every $c \in \mathbb{R}$. We show that f is quasi-concave.

Take any $x, y \in X$ and $\alpha \in [0, 1]$, and let

$$c = \min\{f(x), f(y)\}.$$

Then $x, y \in U_f(c)$. Since $U_f(c)$ is convex,

$$\alpha x + (1 - \alpha)y \in U_f(c).$$

Therefore,

$$f(\alpha x + (1 - \alpha)y) \geq c = \min\{f(x), f(y)\}.$$

Thus f is quasi-concave. □

The previous proposition gives a useful geometric interpretation of quasi-concavity: every upper level set of the function must be convex. For a function on $\mathbb{R}$, this means that every upper level set is an interval (possibly empty).

Proposition Assume that $\succsim$ is represented by u . Then $\succsim$ is convex if and only if u is quasi-concave.

Proof.

Step 1: If $\succsim$ is convex then u is quasi concave.

Take $x, y \in X$ and a number α to show that u is quasi concave. WLOG, assume that $u(x) \geq u(y)$. Then $\alpha x + (1 - \alpha)y \in \{x \in X | x \succsim y\}$. Thus we have

$$u(\alpha x + (1 - \alpha)y) \geq u(y) = \min\{u(x), u(y)\}.$$

Step 2: Suppose that u is quasi-concave. We show that $\succsim$ is convex.

Fix $z \in X$. Since u represents $\succsim$,

$$\{x \in X : x \succsim z\} = \{x \in X : u(x) \geq u(z)\} = U_u(u(z)).$$

By the previous proposition, quasi-concavity of u implies that every upper level set of u is convex. In particular, $U_u(u(z))$ is convex. Therefore,

$$\{x \in X : x \succsim z\}$$

is convex for every $z \in X$, and hence $\succsim$ is convex. □

Remark 8. *The existence of quasi-concave utility function does not exclude the possibility that the existence of another concave utility function.*

Although $f(x) = x^3$ is not concave, consider the strictly increasing transformation

$$\phi(a) = -\exp(-a^{1/3}).$$

Then

$$u(x) = \phi(f(x)) = -e^{-x},$$

which is strictly concave. Thus, the existence of a non-concave quasi-concave representation does not imply that the preference admits no concave representation. The next remark shows that a concave representation need not exist.

Remark 9. *The above remark does not mean that it is always possible to find a concave utility function for convex preferences. The convexity of a preference relation does not necessarily implies an existence of a concave utility function, even though a quasi-concave utility always exists. In particular, we give an example of convex set X and a complete, transitive, and convex relation $\succsim$ such that for every representation u , the function u is not concave, that is, there exist some points $x, y \in X$ and $\alpha \in (0, 1)$ such that*

$$u(\alpha x + (1 - \alpha)y) < \alpha u(x) + (1 - \alpha)u(y).$$

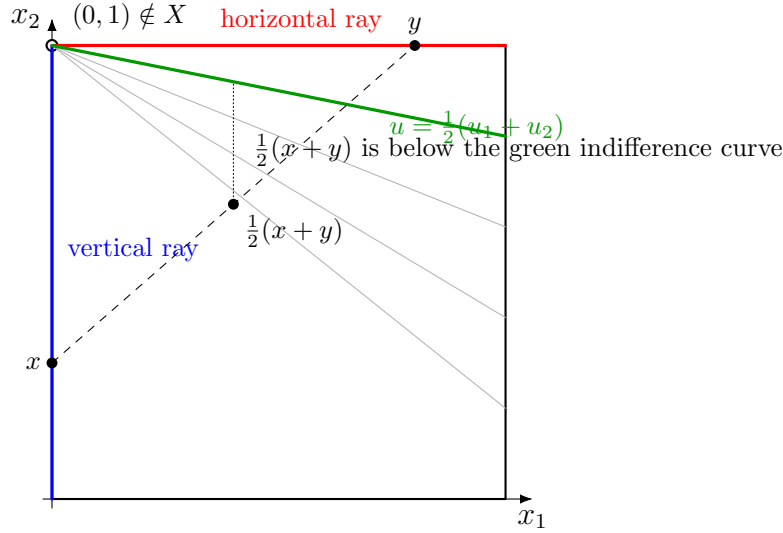
Proof. Let $X = ([0, 1] \times [0, 1]) \setminus \{(0, 1)\}$. The set is convex. The indifference curves are rays emerging from the point $(0, 1)$, and the utility increases as a point moves to the northeast direction. Thus $\succsim$ is convex.

Fix any utility representation u . Denote by u_1 the representation for points on the vertical ray (blue), and by u_2 the representation for points on the horizontal ray (red). Choose $\alpha = \frac{1}{2}$. Find the indifference curve (green) whose utility level is $\frac{1}{2}(u_1 + u_2)$. Notice that this indifference curve may vary depending on the chosen representation u .

It is easy to find a point x on the vertical ray and a point y on the horizontal ray such that the convex combination $\frac{1}{2}(x + y)$ lies below the green indifference curve. That is,

$$u\left(\frac{1}{2}(x + y)\right) < \frac{1}{2}(u(x) + u(y)).$$

Thus, u is not concave. Importantly, such points x and y can be found for every possible representation u of $\succeq$. Therefore, $\succeq$ does not admit a concave representation. $\square$



B: Demand Function

5 Optimal Consumption: Geometric Approach

The consumer's problem is to choose the bundle $x \in X$ that maximizes utility subject to the budget constraint. With N goods at prices $p = (p_1, \dots, p_N)$ and income $I > 0$, the problem is

$$\max_{x \in X} u(x) \quad \text{subject to} \quad p \cdot x \leq I.$$

In the two-good case, the budget line $p_1x_1 + p_2x_2 = I$ can be written as

$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{I}{p_2},$$

a line with slope $-p_1/p_2$ and vertical intercept I/p_2 . The consumer maximizes utility by reaching the highest attainable indifference curve subject to the budget constraint. At an interior optimum, the indifference curve is tangent to the budget line, giving the condition:

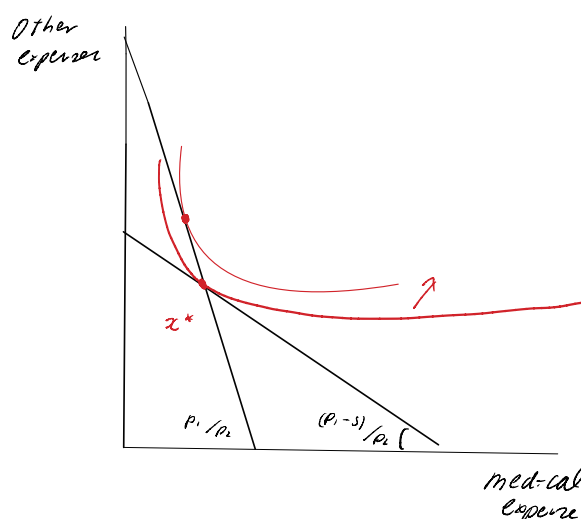
Tangency condition (geometric optimality)

At an interior optimum x^* ,

$$MRS_{12}(x^*) = \frac{p_1}{p_2}.$$

The marginal rate of substitution equals the price ratio. Intuitively: if $MRS_{12} > p_1/p_2$, the consumer values good 1 more than the market does, so it is optimal to consume more of good 1 (and less of good 2) until equality is restored.

5.1 Example: Subsidy to medical expenses of elderly people.



In Japan, people older than 70 pay only 10% of their medical expenses; the other 90% is paid by the government (tax payer).

Is there alternative policy that costs less and does not make the elderly person unhappy.

Let x_1 denote medical expenditure and x_2 all other expenditure. Under a per-unit subsidy of S per unit of medical expense, the elderly consumer faces the subsidised budget constraint

$$(p_1 - S)x_1 + p_2x_2 = I.$$

Let $x^* = (x_1^*, x_2^*)$ be the optimal bundle under the subsidy: $x^* \in \arg \max u(x_1, x_2)$ subject to $(p_1 - S)x_1 + p_2x_2 = I$. The government spends Sx_1^* on the subsidy.

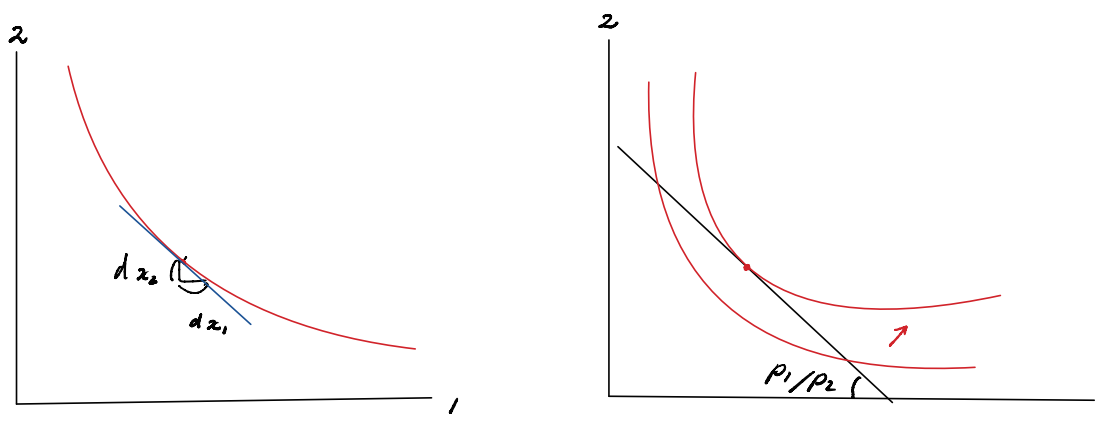
Now consider an alternative policy: abolish the subsidy and instead transfer the cash amount Sx_1^* directly to the consumer. The consumer's new budget set is $\{(x_1, x_2) : p_1x_1 + p_2x_2 \leq I + Sx_1^*\}$, which contains the subsidised optimum x^* (since $p_1x_1^* + p_2x_2^* = I + Sx_1^*$) and also offers bundles previously unaffordable. The consumer is therefore weakly better off under the cash transfer, at the same government cost.

Efficiency of cash transfers over price subsidies

Distorting market prices (via a subsidy or tax) leads to a smaller total welfare than an equivalent lump-sum cash transfer. A free-market allocation maximizes total welfare; price distortions introduce a deadweight loss.

Remark 10 (Equity considerations). *The efficiency argument for cash transfers does not address inequality. How to distribute the total welfare gain involves value judgements that economics, as a positive science, does not resolve. Also, to make the cash transfer, we need to make the elderly person correctly report one's medical spending. However, the elderly person has an incentive to over-report the spending. This is a topic studied in game theory.*

6 Optimal consumption via marginal analysis



Consider the consumer's problem

$$\max_{x \in \mathbb{R}_+^N} u(x) \quad \text{subject to} \quad p \cdot x \leq I,$$

where $p \gg 0$ and $I > 0$. Throughout this section, suppose that u is differentiable and strictly increasing.

Step 1: the budget constraint is binding. At an optimum x^* , we must have

$$p \cdot x^* = I.$$

Indeed, if $p \cdot x^* < I$, then a sufficiently small increase in any good remains feasible and, by strict monotonicity, raises utility. Thus the consumer's problem can be treated as an equality-constrained problem, together with the nonnegativity restrictions $x_i \geq 0$.

Lagrange multiplier theorem. Let $f, g : \mathbb{R}^N \rightarrow \mathbb{R}$ be continuously differentiable. Suppose that x^* is a local maximizer or minimizer of $f(x)$ subject to $g(x) = c$, and that $\nabla g(x^*) \neq 0$. Then there exists a scalar λ such that

$$\nabla f(x^*) = \lambda \nabla g(x^*).$$

Equivalently, x^* satisfies the first-order conditions of the Lagrangian

$$L(x, \lambda) = f(x) + \lambda(c - g(x)).$$

This theorem treats an equality constraint. By itself, it does not account for inequality constraints such as $x_i \geq 0$.

Proposition 5 (Lagrangian first-order conditions). *Consider an interior optimum $x^* \gg 0$ of the consumer's problem. Then the budget constraint is binding and there exists a scalar $\lambda \geq 0$ such that, for every good i ,*

$$MU_i(x^*) = \frac{\partial u(x^*)}{\partial x_i} = \lambda p_i.$$

If the marginal utilities are positive, then $\lambda > 0$ and hence

$$\frac{MU_i(x^*)}{p_i} = \lambda \quad \text{for every } i.$$

Proof. Strict monotonicity implies that the budget constraint is binding, so $p \cdot x^* = I$. Since $x^* \gg 0$, the nonnegativity constraints are locally irrelevant. Apply the Lagrange multiplier theorem to the equality constraint $p \cdot x = I$. Since its gradient is $p \neq 0$, there exists λ such that

$$\nabla u(x^*) = \lambda p,$$

which gives the stated first-order conditions. Monotonicity implies $MU_i(x^*) \geq 0$, so $\lambda \geq 0$; if the marginal utilities are positive, then $\lambda > 0$. $\square$

Two-good case. When $N = 2$ and $MU_1(x^*), MU_2(x^*) > 0$, dividing the two first-order conditions gives

$$MRS_{12}(x^*) = \frac{MU_1(x^*)}{MU_2(x^*)} = \frac{p_1}{p_2}.$$

Equivalently,

$$\frac{MU_1(x^*)}{p_1} = \frac{MU_2(x^*)}{p_2} = \lambda.$$

Thus, at an interior optimum, the marginal utility generated by the last dollar spent is equalized across goods.

Remark 11 (Law of equal marginal utility (per dollar)). *The term $\frac{MU_i(x)}{p_i}$ is the marginal utility gained by spending one more unit of currency on good i . At an interior optimum, the last dollar spent must yield the same marginal utility regardless of which good it is spent on. Notice that*

$$\frac{\partial u / \partial x_i}{p_i} = \underbrace{MU_i(x)}_{\text{(marginal utility of good } i\text{)}} \times \underbrace{\frac{1}{p_i}}_{\text{(the amount of goods } i \text{ can be purchased by one unit of currency)}}$$

is the additional marginal utility by increasing the spending for consumption of good i marginally.

6.1 How to handle the nonnegativity constraints.

The elementary Lagrange multiplier theorem does not directly incorporate the restrictions $x_i \geq 0$. The original consumer problem can nevertheless be solved systematically as follows.

1. **Use monotonicity to impose the binding budget constraint.** Replace $p \cdot x \leq I$ by $p \cdot x = I$.
2. **First look for an interior solution.** Temporarily suppose $x_i > 0$ for every i , apply the Lagrange multiplier theorem, and solve the first-order conditions together with $p \cdot x = I$.
3. **Check the candidate and then check the boundary if necessary.** If the candidate satisfies $x_i > 0$ for all i , it is an interior candidate. One must still verify that it is a maximum; for example, concavity of u makes the first-order conditions sufficient. If the interior candidate is infeasible, or if a corner optimum is possible, set one or more goods equal to zero, solve the resulting lower-dimensional problem on each relevant boundary face, and compare utility across all feasible candidates.

For two goods, the binding budget line has only two corner points,

$$\left(\frac{I}{p_1}, 0\right) \quad \text{and} \quad \left(0, \frac{I}{p_2}\right).$$

Thus one can compare these two corner utilities with the utility at any interior candidate. For N goods, the same idea applies recursively to the faces obtained by setting subsets of goods equal to zero. The Kuhn–Tucker conditions provide a theorem that treats all of these inequality constraints simultaneously, but the step-by-step procedure above is sufficient for the examples considered here.

6.2 Numerical example: Cobb–Douglas utility function

Consider

$$u(x, y) = x^a y^b, \quad a, b > 0,$$

with prices $p_x, p_y > 0$ and income $I > 0$.

Step 1: the budget constraint is binding. The Cobb–Douglas preference is locally nonsatiated, so the consumer spends all income. More directly, if the budget had slack, one could make a sufficiently small feasible increase in the goods that raises utility. Hence

$$p_x x + p_y y = I.$$

Step 2: solve for the interior candidate. The Lagrangian is

$$L(x, y, \lambda) = x^a y^b + \lambda(I - p_x x - p_y y).$$

The first-order conditions for an interior optimum are

$$a x^{a-1} y^b = \lambda p_x, \quad b x^a y^{b-1} = \lambda p_y, \quad p_x x + p_y y = I.$$

Dividing the first condition by the second gives

$$\frac{a}{b} \frac{y}{x} = \frac{p_x}{p_y},$$

or equivalently,

$$\frac{p_x x}{p_y y} = \frac{a}{b}.$$

Combining this with the budget constraint yields

$$x^*(p, I) = \frac{a}{a+b} \frac{I}{p_x}, \quad y^*(p, I) = \frac{b}{a+b} \frac{I}{p_y}.$$

Thus the consumer allocates the constant expenditure share $a/(a+b)$ to good x and $b/(a+b)$ to good y .

Step 3: verify the interior solution and rule out corners. The candidate satisfies $x^* > 0$ and $y^* > 0$. Moreover, the two corner points give

$$u\left(\frac{I}{p_x}, 0\right) = 0, \quad u\left(0, \frac{I}{p_y}\right) = 0,$$

whereas $u(x^*, y^*) > 0$. Hence neither corner can be optimal.

Finally, because the logarithm is strictly increasing, maximizing $u(x, y)$ over $x, y > 0$ is equivalent to maximizing

$$\log u(x, y) = a \log x + b \log y.$$

This transformed objective is strictly concave, so the interior first-order conditions identify the unique global optimum. Therefore the Marshallian demand is

$$\boxed{x^*(p, I) = \frac{a}{a+b} \frac{I}{p_x}, \quad y^*(p, I) = \frac{b}{a+b} \frac{I}{p_y}}.$$

7 Optimal consumption: demand function

Consider a consumer with a continuous, strictly quasi-concave utility function $u : \mathbb{R}_+^N \rightarrow \mathbb{R}$ facing prices $p = (p_1, \dots, p_N) \in \mathbb{R}_{++}^N$ and income $I > 0$.

Definition 7.1. A function f is *strictly quasi-concave* if for any $x, y \in X$ and $\alpha \in [0, 1]$ such that

$$f(\alpha x + (1 - \alpha)y) > \min\{f(x), f(y)\}.$$

The consumer's problem is

$$\max_{x \in \mathbb{R}_+^N} u(x) \quad \text{subject to} \quad p \cdot x \leq I. \quad (1)$$

Under the maintained assumptions, the solution to (1) is unique for every (p, I) .

Definition 7.2 (Marshallian demand function). The Marshallian (or ordinary) demand function is the mapping

$$x^* : \mathbb{R}_{++}^N \times \mathbb{R}_{++} \rightarrow \mathbb{R}_+^N, \quad (p, I) \mapsto x^*(p, I),$$

where $x^*(p, I)$ is the unique solution to (1).

After we learn two basic properties of the demand function, we study how demand changes with respect to income and prices.

7.1 Basic properties (MWG §3.D)

Two fundamental restrictions on demand follow immediately from the structure of the UMP.

Proposition 6 (Homogeneity of degree zero (MWG Prop. 3.D.2)). *If $x^*(p, I)$ is derived from utility maximization, then for all (p, I) and all $\lambda > 0$,*

$$x^*(\lambda p, \lambda I) = x^*(p, I).$$

Proof. The budget set at $(\lambda p, \lambda I)$ is $\{x \in \mathbb{R}_+^N : \lambda p \cdot x \leq \lambda I\} = \{x \in \mathbb{R}_+^N : p \cdot x \leq I\}$, which is identical to the budget set at (p, I) . Since the feasible set is unchanged, the utility-maximising bundle is unchanged. $\square$

No money illusion

Homogeneity of degree zero says that only relative prices and real income matter. Doubling all prices and income simultaneously leaves the budget set — and hence optimal demand — exactly unchanged. This rules out money illusion: a rational consumer does not change behaviour when all nominal values scale by the same factor.

A practical consequence: differentiating $x_i^*(\lambda p, \lambda I) = x_i^*(p, I)$ for each i with respect to λ at $\lambda = 1$ (Euler's theorem) gives

$$\sum_{j=1}^N \frac{\partial x_i^*}{\partial p_j} p_j + \frac{\partial x_i^*}{\partial I} I = 0 \quad \text{for each } i.$$

Proposition 7 (Walras' law (MWG Prop. 3.D.3)). *If u is monotonically increasing and $x^*(p, I)$ solves the consumer problem, then*

$$p \cdot x^*(p, I) = I.$$

7.2 How demand changes as income and (own) price change

7.2.1 Income Effects

Holding prices fixed, an increase in income shifts the budget set outward. The qualitative response of demand to this shift classifies goods as follows.

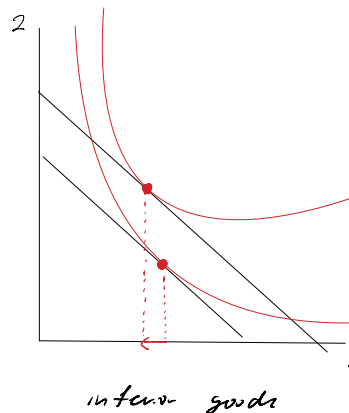
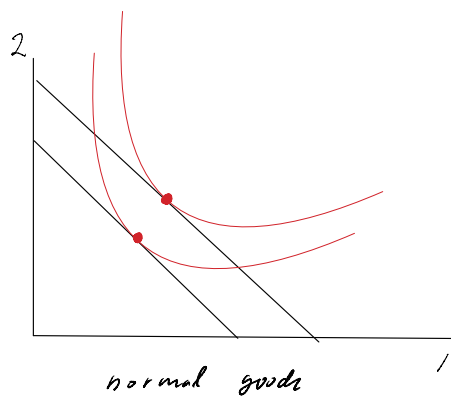
Definition 7.3 (Normal and inferior goods).

Good i is a normal good at (p, I) if

$$\frac{\partial x_i^*(p, I)}{\partial I} > 0,$$

and an inferior good at (p, I) if

$$\frac{\partial x_i^*(p, I)}{\partial I} < 0.$$



Remark 12. *In the Cobb–Douglas example, every good is normal: $\partial x_i^* / \partial I = a_i / (\sum_j a_j) \cdot 1/p_i > 0$.*

7.2.2 Price Effect (Own-Price Effects)

We now study how the demand changes with respect to prices. We first study how the demand of a good changes with respect to its own price.

Typically, we observe the following:

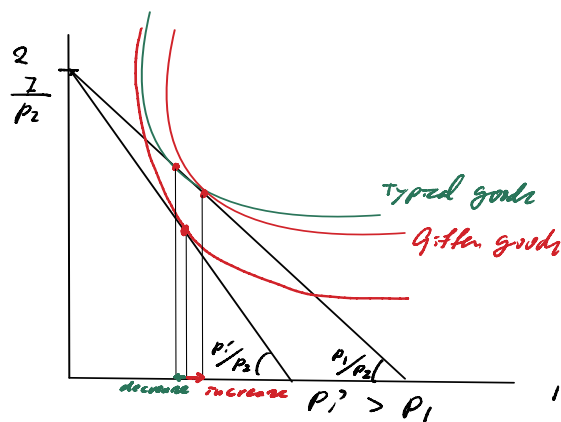
$$\frac{\partial x_i^*(p, I)}{\partial p_i} < 0.$$

If it holds, we say that good i satisfies the law of demand at (p, I) .

There are some cases in which the above does not hold:

Definition 7.4 (Giffen good). Good i is a Giffen good at (p, I) if

$$\frac{\partial x_i^*(p, I)}{\partial p_i} > 0.$$



Remark 13 (Giffen goods).

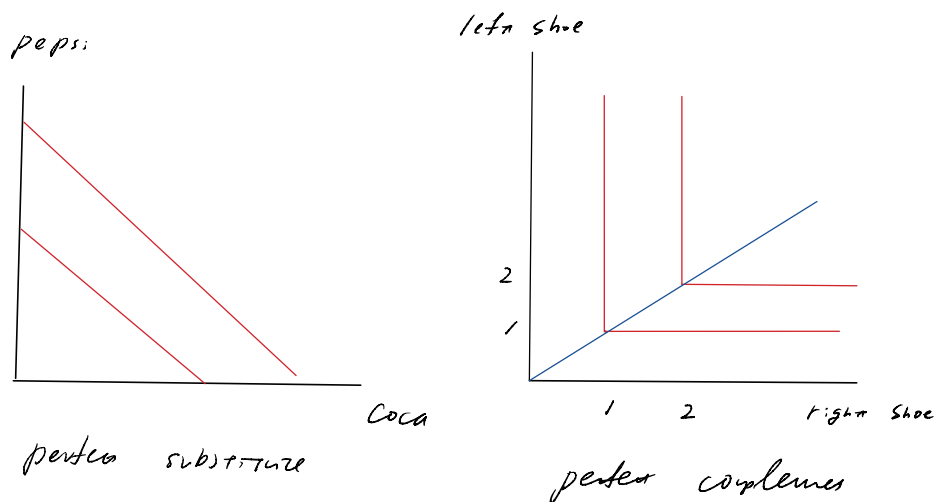
- The law of demand does not hold universally for Marshallian demand. Giffen goods are theoretically possible and have been observed in certain low-income populations.
- Giffen goods are typically inferior staple foods, such as rice, that account for a large share of a poor household's expenditure.
- Consider a very poor household that spends most of its income on rice and a small amount on more expensive foods, such as meat, to obtain protein.
- Suppose the price of rice increases. The household becomes effectively poorer because its purchasing power falls. It may no longer be able to afford as much meat. To obtain enough calories and avoid hunger, the household may reduce its consumption of meat and increase its consumption of rice, even though rice has become more expensive.

- *This example illustrates that a price increase has two effects: a substitution effect, which reduces demand for rice as it becomes relatively more expensive, and an income effect, which increases demand for rice because rice is an inferior good. When the income effect dominates the substitution effect, the good is Giffen. We will study these two effects in the following sections.*

8 Compensated demand function and expenditure function—tools to study substitution effect

8.1 Motivation (Heuristic)

When the price of one good changes, two conceptually distinct forces act on demand: a substitution effect (the consumer re-optimises along the same indifference curve) and an income effect (the consumer's real purchasing power changes). To isolate the substitution effect, we introduce the compensated (Hicksian) demand function.



Recall the polar cases of perfect substitutes and perfect complements. For most goods, indifference curves lie between these extremes, and a key question is how strongly two goods substitute for or complement each other. The natural tool for measuring this is the compensated (Hicksian) demand function.

Given prices p and a target utility level u , the compensated demand $x^h(p, u)$ is the solution to the expenditure minimisation problem:

Expenditure Minimisation Problem/Compensated (Hicksian) Demand

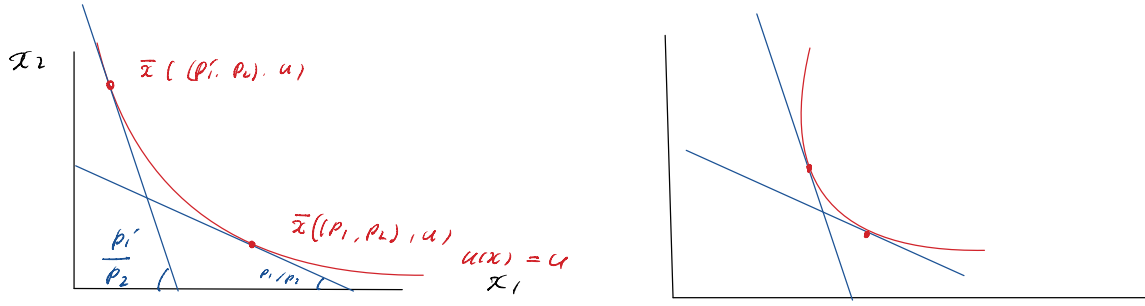
$$x^h(p, u) = \arg \min_x p \cdot x \quad \text{subject to} \quad u(x) \geq u.$$

$x^h(p, u)$ answers: “what is the cheapest bundle that achieves utility u at prices p ?”

To measure substitutability, consider a two-good economy ($N = 2$) and a price increase of good 1 from p_1 to p'_1 . The compensated demands $x^h(p_1, p_2, u)$ and $x^h(p'_1, p_2, u)$ both keep utility fixed, so any change in x^h_2 reflects pure substitution.

When the two goods are substitutes, $x^h(p_1, p_2, u)$ and $x^h(p'_1, p_2, u)$ differ substantially (the consumer switches toward good 2 when good 1 becomes more expensive).⁴

⁴In the figure, I write $\bar{x}$ instead of x^h .



When they are complements, the two compensated demands are close (the consumer uses both goods together). A precise measure is the cross-price derivative of compensated demand:

- if $\partial x_2^h(p_1, p_2, u)/\partial p_1$ is large and positive, the goods are strong substitutes;
- if it is negative, they are complements. (Formalized in the next section.)

Note also that $\partial x_1^h(p_1, p_2, u)/\partial p_1 < 0$ in the figure:

- When the own price rises, compensated demand for the good itself falls. This is the negativity of the own-substitution effect, proved formally in Theorem 3.

8.2 Compensated demand and expenditure function

Definition 8.1 (Compensated demand and expenditure function). Given prices $p \in \mathbb{R}_{++}^N$ and a target utility level $\bar{u}$, define

$$e(p, \bar{u}) \equiv \min_{x \in \mathbb{R}_+^N} p \cdot x \quad \text{subject to} \quad u(x) \geq \bar{u}. \quad (2)$$

The function $e : \mathbb{R}_{++}^N \times \mathbb{R} \rightarrow \mathbb{R}_+$ is the expenditure function, and the solution

$$x^h(p, \bar{u}) \equiv \arg \min_{x \in \mathbb{R}_+^N} \{p \cdot x : u(x) \geq \bar{u}\}$$

is the compensated (Hicksian) demand function.

The expenditure function and compensated demand are linked by

$$e(p, \bar{u}) = p \cdot x^h(p, \bar{u}).$$

Remark 14. The expenditure function measures the minimum cost of attaining utility $\bar{u}$ at prices p . It is the natural tool for welfare analysis: the change $e(p', \bar{u}) - e(p, \bar{u})$ quantifies the monetary impact of a price change from p to p' , holding welfare constant.

8.3 General Definitions: Substitutes and Complements

Definition 8.2 (Hicksian substitutes and complements). Goods i and j are (Hicksian) substitutes at $(p, \bar{u})$ if

$$\frac{\partial x_j^h(p, \bar{u})}{\partial p_i} > 0,$$

and (Hicksian) complements at $(p, \bar{u})$ if

$$\frac{\partial x_j^h(p, \bar{u})}{\partial p_i} < 0.$$

Intuitively: if the price of good i rises (with utility held constant), and the consumer responds by increasing compensated demand for good j , then j is a substitute for i (the consumer switches toward j). If compensated demand for j falls, the goods are complements (the consumer reduces j along with i).

Remark 15.

- The relation that goods i and j are complements are symmetric relations.

$$\frac{\partial x_j^h(p, \bar{u})}{\partial p_i} > 0 \implies \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} > 0.$$

- The relation is not transitive. Even if goods i and j are complements; and j and k are complements; it is possible that i and k are substitutes.
- The definitions above make sense for $N \geq 3$. When $N = 2$, all goods are necessarily substitutes in the Hicksian sense. To see this: if p_1 rises, negativity of self-substitution (Theorem 3) gives $\partial x_1^h / \partial p_1 \leq 0$, so the consumer demands less of good 1. But utility must be maintained, so compensated demand for good 2 must increase, i.e. $\partial x_2^h / \partial p_1 > 0$. Complementarity requires $N \geq 3$.

8.4 Negativity of the Own-Substitution Effect (Compensated Law of Demand)

The following result shows that compensated demand always satisfies the law of demand.

Theorem 3 (Negativity of own-substitution effect). For every good i ,

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} \leq 0.$$

Proof. Fix a utility level $\bar{u}$. Let p^0 and p^1 be two price vectors, and let $x^0 = x^h(p^0, \bar{u})$ and $x^1 = x^h(p^1, \bar{u})$ be the corresponding compensated demands.

Step 1. By definition of expenditure minimisation, x^1 minimises cost at prices p^1 subject to $u(x) \geq \bar{u}$. Since x^0 is feasible (it also achieves $\bar{u}$), we have

$$p^1 \cdot x^1 \leq p^1 \cdot x^0.$$

Step 2. By the symmetric argument (exchanging the roles of p^0 and p^1),

$$p^0 \cdot x^0 \leq p^0 \cdot x^1.$$

Step 3. Subtracting the second inequality from the first:

$$(p^1 - p^0) \cdot x^1 = p^1 \cdot x^1 - p^0 \cdot x^1 \leq p^1 \cdot x^0 - p^0 \cdot x^0 = (p^1 - p^0) \cdot x^0,$$

which rearranges to

$$(p^1 - p^0) \cdot (x^1 - x^0) \leq 0. \quad (3)$$

Step 4. Suppose the two price vectors differ only in good i , with $p_i^1 > p_i^0$. Then (3) reduces to

$$(p_i^1 - p_i^0)(x_i^{h1} - x_i^{h0}) \leq 0.$$

Since $p_i^1 - p_i^0 > 0$, we conclude $x_i^{h1} - x_i^{h0} \leq 0$, and hence $\partial x_i^h / \partial p_i \leq 0$. $\square$

Remark 16. *The inequality (3) is a general property of compensated demand: $\Delta p \cdot \Delta x \leq 0$. This is precisely analogous to the law of demand for factor inputs in production theory.*

8.5 Properties of Expenditure Function

Definition 8.3 (Expenditure function). The expenditure function $e : \mathbb{R}_{++}^N \times \mathbb{R} \rightarrow \mathbb{R}_+$ is defined by

$$e(p, \bar{u}) = \min_{x \in \mathbb{R}_+^N} \{p \cdot x : u(x) \geq \bar{u}\} = p \cdot x^h(p, \bar{u}).$$

The expenditure function is the key tool for measuring welfare gains and losses from price changes.

Example 6 (Trans-Pacific Partnership (TPP)). *The U.S. is considering joining the TPP, which would change the prices faced by domestic farmers from p to p' . A farmer currently at utility level u would be harmed if prices change adversely. The government wishes to compensate the farmer just enough to restore utility u . The required subsidy is exactly $e(p', u) - e(p, u)$, the change in the expenditure function.*

Proposition 8 (Properties of the expenditure function). *Under the maintained assumptions on u , the expenditure function satisfies:*

- (a) **Non-decreasing in p :** $p' \geq p \implies e(p', \bar{u}) \geq e(p, \bar{u})$.
- (b) **Homogeneous of degree 1 in p :** $e(tp, \bar{u}) = t e(p, \bar{u})$ for all $t > 0$.
- (c) **Concave in p :** For all $\lambda \in [0, 1]$,

$$e(\lambda p + (1 - \lambda)p', \bar{u}) \geq \lambda e(p, \bar{u}) + (1 - \lambda) e(p', \bar{u}).$$

- (d) **Strictly increasing in $\bar{u}$** (provided utility is locally non-satiated).

Proof. Statements (a), (b), (c) holds trivially. (d) follows from the envelope theorem. Fix $\lambda \in [0, 1]$ to show (c). For any $\tilde{p}$ and u , let $x^h(\tilde{p}, u)$ be the solutions of the expenditure minimization problem. Then

$$\begin{aligned} e(\lambda p + (1 - \lambda)p', u) &= (\lambda p + (1 - \lambda)p') \cdot x^h(\lambda p + (1 - \lambda)p', u) \\ &= \lambda(p \cdot x^h(\lambda p + (1 - \lambda)p', u)) + (1 - \lambda)(p' \cdot x^h(\lambda p + (1 - \lambda)p', u)) \\ &\geq \lambda(p \cdot x^h(p, u)) + (1 - \lambda)(p' \cdot x^h(p', u)) \\ &= \lambda e(p, u) + (1 - \lambda)e(p', u). \end{aligned}$$

□

8.6 Shephard's Lemma—Changes in real income induced by price changes

Even when nominal income I is held fixed, a change in prices alters the consumer's real income—i.e., the set of affordable bundles. We make this precise via the expenditure function.

Proposition 9 (Shephard's Lemma). *Suppose the expenditure function $e(p, \bar{u})$ is differentiable in p . Then, for each good i ,*

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}).$$

Proof. Fix p and $\bar{u}$, and let $x^h = x^h(p, \bar{u})$.

Consider a change in the price of good i from p_i to $p_i + h$, holding all other prices fixed. Let $\mathbf{e}_i$ denote the i -th unit vector.

Since x^h remains feasible after the price change, the minimum expenditure at the new prices cannot exceed the cost of purchasing x^h :

$$e(p + h\mathbf{e}_i, \bar{u}) \leq (p + h\mathbf{e}_i) \cdot x^h = e(p, \bar{u}) + hx_i^h.$$

Thus, for $h > 0$,

$$\frac{e(p + h\mathbf{e}_i, \bar{u}) - e(p, \bar{u})}{h} \leq x_i^h,$$

whereas for $h < 0$, the inequality is reversed.

Since e is differentiable at p , the limits from both sides must coincide. Therefore,

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h.$$

□

Remark 17 (Intuition and the Envelope Theorem). *Shephard's Lemma illustrates a general principle of optimization: when a parameter changes, the adjustment of the optimal choice has no first-order effect on the optimized value.*

To see this, suppose that Hicksian demand is differentiable and the solution is interior. Differentiating $e(p, \bar{u}) = p \cdot x^h(p, \bar{u})$ gives

$$\frac{\partial e}{\partial p_i} = \underbrace{x_i^h}_{\text{Direct effect}} + \underbrace{p \cdot \frac{\partial x^h}{\partial p_i}}_{\text{Indirect effect}}.$$

The first term is the direct effect of the price change. The second term is the indirect effect through the adjustment of Hicksian demand.

Since Hicksian demand minimizes expenditure subject to a fixed utility level, the first-order condition implies that expenditure does not change to first order when the bundle moves along the indifference curve. Thus, under the usual regularity conditions, $p \cdot (\partial x^h / \partial p_i) = 0$. Consequently, only the direct effect remains. This logic holds more generally and is known as the envelope theorem. Consider

$$V(t) = f(x^*(t), t),$$

where $x^*(t)$ is an interior maximizer or minimizer of $f(x, t)$. Assuming differentiability, the chain rule gives

$$\frac{dV(t)}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{\text{Direct effect}} + \underbrace{\nabla_x f \cdot \frac{dx^*(t)}{dt}}_{\text{Indirect effect}=0},$$

where the derivatives of f are evaluated at $(x^*(t), t)$. The indirect effect vanishes because the first-order condition implies $\nabla_x f = 0$ at the optimum. Thus, to differentiate an optimized value function, we need only calculate the direct effect of the parameter, holding the optimal choice fixed.

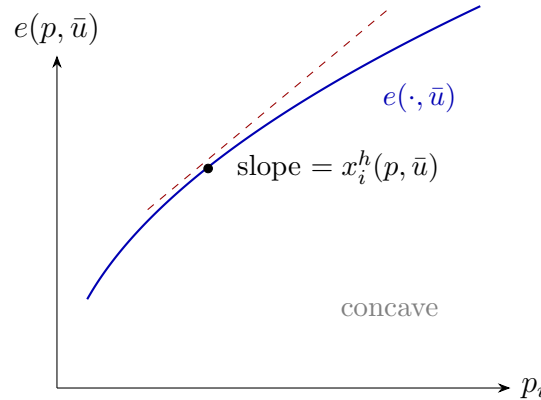


Figure 4: The expenditure function is concave in prices. The slope at any point equals the compensated demand (Shephard's lemma).

Example 7 (Water Consumption). Suppose a consumer purchases $x_i^* = 4$ bottles of water per week. If the per-bottle price increases by $\Delta p_i = \$1$, the consumer needs approximately $\Delta I = 4 \times 1 = \$4$ of additional income to maintain the original utility level.

Note that with an additional \$4 in income, the consumer could in principle re-optimize and potentially achieve a strictly higher utility level (since \$4 is a sufficient but not necessary compensation at the new prices). Shepard's lemma shows locally this effect is small.

9 Slutsky Equation: how demand changes as prices change

We study the mechanism by which demand for a good changes in response to a change in its own price. Throughout, we work in an n -good economy with price vector $p = (p_1, \dots, p_n) \in \mathbb{R}_{++}^n$ and income $I > 0$. The consumer has a continuous, strictly quasi-concave, and locally nonsatiated utility function $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$.

For this purpose, we define an *indirect utility function* as follows:

Definition 9.1 (Indirect utility function). The **indirect utility function** is the value function of the UMP:

$$v(p, I) \equiv \max_{x \geq 0} u(x) \quad \text{subject to} \quad p \cdot x \leq I.$$

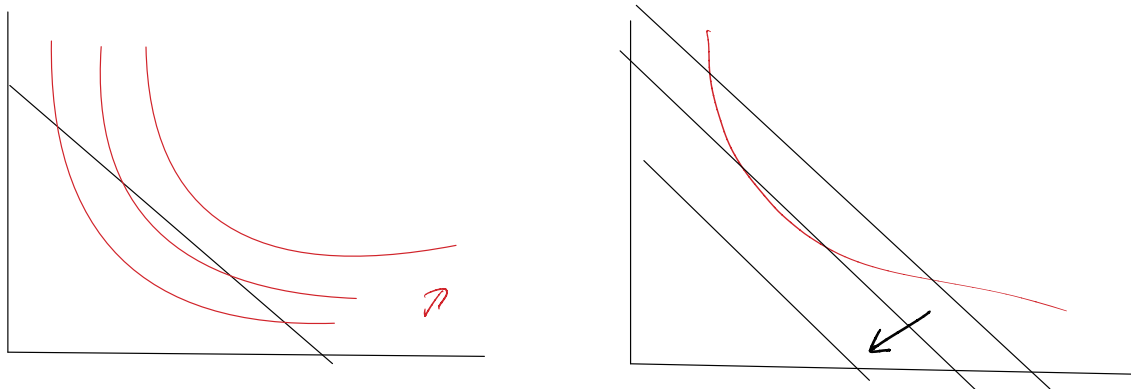
It records the maximized utility level attainable at prices $p \in \mathbb{R}_{++}^L$ and income $I > 0$.

9.1 Duality of the consumer's problem

Under our maintained assumptions on u , the following duality relations hold.

Proposition 10 (Duality). *Let x^* be the optimal bundle at (p, I) . Then:*

1. $x(p, I) = x^* = x^h(p, \bar{u})$, where $\bar{u} = v(p, I)$, (Marshallian = Hicksian at the optimum);
2. $e(p, v(p, I)) = I$ (expenditure recovers income);
3. $v(p, e(p, \bar{u})) = \bar{u}$, for any $\bar{u}$ (indirect utility recovers the target utility).



Remark 18. Whether we frame the consumer's problem as "maximize utility given a budget" or "minimize cost to achieve a target utility," we arrive at the same optimal bundle. This duality is the engine behind the Slutsky decomposition.

9.2 The Slutsky Equation

We now derive the fundamental decomposition of the price effect into substitution and income components.

Theorem 4 (Slutsky Equation). *Let $\bar{u} = v(p, I)$. Then for each good i and price p_i ,*

$$\frac{\partial x_i(p, I)}{\partial p_i} = \underbrace{\frac{\partial x_i^h(p, \bar{u})}{\partial p_i}}_{\text{Substitution Effect (SE)}} - \underbrace{\frac{\partial x_i(p, I)}{\partial I} x_i^*}_{\text{Income Effect (IE)}}. \quad (4)$$

Proof. By duality (Proposition 10), for all p and corresponding $\bar{u} = v(p, I)$:

$$x_i^h(p, \bar{u}) = x_i(p, e(p, \bar{u})).$$

Differentiating both sides with respect to p_i (applying the chain rule on the right-hand side):

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, e(p, \bar{u}))}{\partial p_i} + \frac{\partial x_i(p, e(p, \bar{u}))}{\partial I} \cdot \frac{\partial e(p, \bar{u})}{\partial p_i}. \quad (5)$$

By Shephard's Lemma (Proposition 9),

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}) = x_i^*.$$

Substituting into (5) and noting $e(p, \bar{u}) = I$:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, I)}{\partial p_i} + \frac{\partial x_i(p, I)}{\partial I} x_i^*.$$

Rearranging yields the Slutsky equation (4). □

Interpretation of the Slutsky Equation

The total effect of an own-price increase on the Marshallian demand decomposes as:

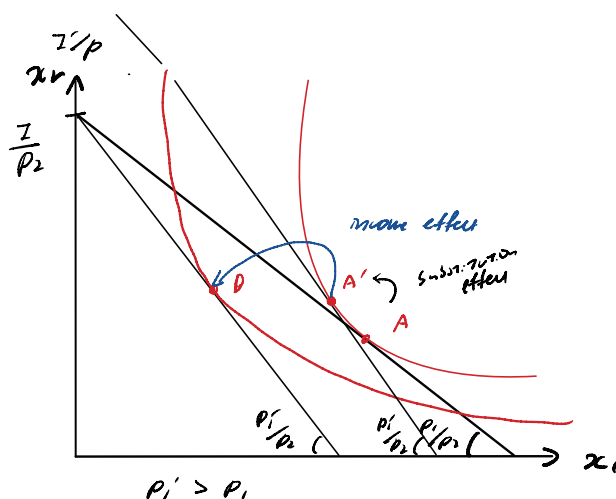
$$\underbrace{\frac{\partial x_i(p, I)}{\partial p_i}}_{\text{Total Effect}} = \underbrace{\frac{\partial x_i^h(p, \bar{u})}{\partial p_i}}_{\text{SE: } \leq 0} - \underbrace{\frac{\partial x_i(p, I)}{\partial I} x_i^*}_{\text{IE}}.$$

- **Substitution Effect (SE):** The change in demand holding utility constant (movement along the indifference curve). By the negativity of self-substitution effect, $\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} \leq 0$ always. That is, compensated demand curves always slope downward.
- **Income Effect (IE):** The change in demand due to the reduction in real income caused by the price increase, multiplied by the original quantity consumed.
 - If good i is a normal good ($\frac{\partial x_i}{\partial I} \geq 0$): The IE is negative (reinforces the SE).

- If good i is an inferior good ($\frac{\partial x_i}{\partial I} < 0$): The IE is positive (offsets the SE).
- If good i is a Giffen good: The IE dominates the SE, so $\frac{\partial x_i}{\partial p_i} > 0$.

Remark 19. The general (cross-price) Slutsky equation for the effect of p_j on x_i takes the form

$$\frac{\partial x_i(p, I)}{\partial p_j} = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} - \frac{\partial x_i(p, I)}{\partial I} x_j^*.$$



Remark 20 (Giffen goods: an intuitive “poor household staple” story). A Giffen good is a good whose Marshallian demand increases when its own price increases:

$$\frac{\partial x_1(p, w)}{\partial p_1} > 0.$$

The standard mechanism is a very large negative income effect for a staple that is strongly inferior and takes a large budget share.

A canonical example (with supportive empirical evidence in very poor populations) is a basic staple such as rice. When the price of rice rises, the household’s real income falls: at the old consumption bundle they can no longer afford the same diet, so they become effectively poorer. For a sufficiently poor household, “better” foods (meat, eggs, vegetables, etc.) are the marginal items that get cut first. To maintain calories, the household substitutes away from these higher-quality foods and ends up relying more on the cheap staple. In that case the income effect of the rice price increase raises rice consumption so much that it outweighs the substitution effect (which always pushes rice consumption down holding utility fixed). The net result is the Giffen pattern: higher rice prices can lead to higher rice purchases.

10 Price/Income Elasticity of Demand

Having established how demand responds to prices, we now ask the quantitative question: by how much does demand change?

In this question, we ask quantitative question: How much demand changes with respect to the change of prices?

10.1 Why Elasticity? A Unit-Free Measure of Responsiveness

A natural first measure of how demand responds to price is $\Delta x / \Delta p$. But this ratio depends on the units chosen: if we measure beer in millilitres rather than litres, the ratio changes by a factor of 1000. To obtain a unit-free measure, we express both changes as percentages. Since $\Delta x / x$ is invariant to rescaling (a factor of 1000 in numerator and denominator cancels), the price elasticity of demand is defined as the percentage change in quantity per one-percent change in price:

Price and income elasticities

$$\varepsilon = \frac{\Delta x / x}{\Delta p / p}, \quad \eta = \frac{\Delta x / x}{\Delta I / I}.$$

Both measures are dimensionless and therefore comparable across goods and markets.

10.2 Elasticity: Definition and Motivation

Definition 10.1 (Price Elasticity of Demand). The **(own-price) elasticity of demand** for good i is

$$\varepsilon_{ii} \equiv -\frac{\partial x_i / x_i}{\partial p_i / p_i} = -\frac{\partial x_i}{\partial p_i} \cdot \frac{p_i}{x_i}. \quad (6)$$

The negative sign is a convention so that $\varepsilon_{ii} \geq 0$ for ordinary (non-Giffen) goods.

Remark 21 (Unit-Free Measurement). *A naive measure of price responsiveness, $\Delta x_i / \Delta p_i$, depends on the units chosen for quantity and price. For instance, measuring beer in liters versus milliliters changes the ratio by a factor of 1000. The percentage formulation*

$$\varepsilon_{ii} = -\frac{\Delta x_i / x_i}{\Delta p_i / p_i}$$

is unit-free, since

$$\frac{\Delta x_i / x_i}{\Delta p_i / p_i} = \frac{(1000 \Delta x_i) / (1000 x_i)}{\Delta p_i / p_i},$$

and the scaling factor cancels.

Definition 10.2 (Income Elasticity of Demand). The **income elasticity of demand** for good i is

$$\eta_i \equiv \frac{\partial x_i}{\partial I} \cdot \frac{I}{x_i}.$$

Good i is normal if $\eta_i > 0$; inferior if $\eta_i < 0$; a luxury good if $\eta_i > 1$; and a necessity if $0 < \eta_i < 1$.

10.3 Application: How Revenue Changes with Prices

Consider a firm producing good i with revenue $R = p_i x_i(p, I)$. Differentiating with respect to p_i :

$$\frac{\partial R}{\partial p_i} = x_i + p_i \frac{\partial x_i}{\partial p_i} = x_i \left(1 + \frac{p_i}{x_i} \frac{\partial x_i}{\partial p_i} \right) = x_i (1 - \varepsilon_{ii}). \quad (7)$$

Revenue and Elasticity

$$\frac{\partial R}{\partial p_i} = x_i (1 - \varepsilon_{ii}).$$

- $\varepsilon_{ii} < 1$ (inelastic): A price increase raises revenue.
- $\varepsilon_{ii} = 1$ (unit elastic): Revenue is maximized; a marginal price change has no first-order revenue effect.
- $\varepsilon_{ii} > 1$ (elastic): A price increase lowers revenue.

10.4 Determinants of Price Elasticity: An Elasticity Form of the Slutsky Equation

We now connect the Slutsky equation (Theorem 4) to elasticities, providing a structural account of what makes demand more or less elastic.

Multiply both sides of the Slutsky equation (4) by $-p_i/x_i$:

$$\underbrace{-\frac{\partial x_i(p, I)}{\partial p_i} \frac{p_i}{x_i}}_{\varepsilon_{ii}} = \underbrace{-\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} \frac{p_i}{x_i}}_{\varepsilon_{ii}^h} + \underbrace{\frac{\partial x_i(p, I)}{\partial I} \frac{I}{x_i}}_{\eta_i} \cdot \underbrace{\frac{p_i x_i}{I}}_{w_i}, \quad (8)$$

where:

- ε_{ii} is the **Marshallian price elasticity**;
- ε_{ii}^h is the **compensated (Hicksian) price elasticity**—the elasticity along the indifference curve;
- η_i is the **income elasticity of demand**;
- $w_i = p_i x_i / I$ is the **expenditure (budget) share** of good i .

Determinants of Price Elasticity

The Marshallian price elasticity ε_{ii} is large when:

- (1) **Close substitutes are available** (ε_{ii}^h is large): When the consumer can easily switch to alternative goods, the compensated demand is highly responsive to price changes.
- (2) **Income elasticity is high** (η_i is large): Luxury goods ($\eta_i > 1$) tend to have larger overall elasticities because the income effect amplifies the substitution effect.
- (3) **Budget share is high** (w_i is large): Goods that command a large fraction of the consumer's budget generate a substantial real-income reduction when their price rises, magnifying the income effect.

Example 8 (Empirical Price Elasticities in the United States). *The following table reports estimated own-price elasticities for selected goods, ordered from least to most elastic.*

<i>Good</i>	$\hat{\varepsilon}_{ii}$	<i>Category</i>	<i>Key driver</i>
<i>Eggs</i>	0.1	<i>Inelastic</i>	<i>Few substitutes, small w_i</i>
<i>Beef</i>	0.4	<i>Inelastic</i>	<i>Some substitutes (poultry, pork)</i>
<i>Gasoline</i>	0.5	<i>Inelastic</i>	<i>Necessity, limited short-run substitutes</i>
<i>Housing</i>	1.2	<i>Elastic</i>	<i>Large w_i, some η_i</i>
<i>Restaurant meals</i>	2.3	<i>Elastic</i>	<i>Many substitutes (home cooking), luxury</i>
<i>Airline travel</i>	2.4	<i>Elastic</i>	<i>Substitutes (car, train), high η_i</i>
<i>Foreign travel</i>	4.1	<i>Elastic</i>	<i>High η_i (luxury), many substitutes</i>

The pattern is consistent with the three determinants in equation (8):

- **Eggs** ($\varepsilon = 0.1$): Eggs are a dietary staple with no close substitutes. The budget share w_i is negligible, so the income effect is tiny. Almost all of the (small) response is a substitution effect.
- **Gasoline** ($\varepsilon = 0.5$): In the short run, consumers cannot easily substitute away from gasoline (commute distances are fixed). The budget share is non-negligible, contributing some income effect.
- **Housing** ($\varepsilon = 1.2$): The large budget share (w_i often 25–35%) generates a substantial income effect, pushing elasticity above unity even though substitution options are limited.
- **Foreign travel** ($\varepsilon = 4.1$): This is a luxury good ($\eta_i \gg 1$) with many substitutes (domestic travel, staycations). All three factors—high ε_{ii}^h , high η_i , and moderate w_i —contribute to the large elasticity.

Goods	Is there close substitute?	luxurious goods?	high expenditure share ?
Eggs	Probably No	No	No
Beef	Yes	No	No
Gasoline	No	No	Yes/No
Housing	No	No	Yes
Restaurant meals	Yes	Yes/No	Yes
Airline travel	No	No	Yes
Foreign travel	Yes (e.g., domestic travel)	Yes	Yes

C: How to test rationality from observed demand

11 The Weak Axiom of Revealed Preference (MWG §2.F)

Section 11 (this section) applies the abstract WARP framework of §1.C to the Walrasian demand function $x(p, w)$. WARP in this context has sharp quantitative implications: it is equivalent to a compensated law of demand and implies that the Slutsky substitution matrix is negative semidefinite.

Throughout this section, we assume that $x(p, w)$ satisfies Walras' law: $p \cdot x(p, w) = w$ for all $(p, w) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}$.

11.1 Definition of WARP for Walrasian Demand

Definition 11.1 (WARP for Walrasian demand (MWG Def. 2.F.1)). The demand function $x(p, w)$ satisfies the weak axiom of revealed preference (WARP) if: for any two price–wealth pairs (p, w) and (p', w') ,

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \implies p' \cdot x(p, w) > w'. \quad (9)$$

Economic intuition for WARP

- The bundle $x(p, w)$ is the consumer's choice at budget (p, w) .
- If $x(p', w')$ was affordable at (p, w) (i.e. $p \cdot x(p', w') \leq w$) but was not chosen (the consumer chose the different bundle $x(p, w)$ instead), then $x(p', w')$ is revealed worse than $x(p, w)$.
- WARP then requires that $x(p, w)$ is not affordable at (p', w') (i.e. $p' \cdot x(p, w) > w'$), since otherwise the consumer would be “choosing $x(p', w')$ over $x(p, w)$ even though $x(p, w)$ was available,” contradicting the previous revealed preference.

Remark 22 (Equivalent formulation). *WARP can be restated symmetrically: if $x(p, w)$ is chosen when $x(p', w')$ was affordable, then we cannot have $x(p', w')$ chosen when $x(p, w)$ is*

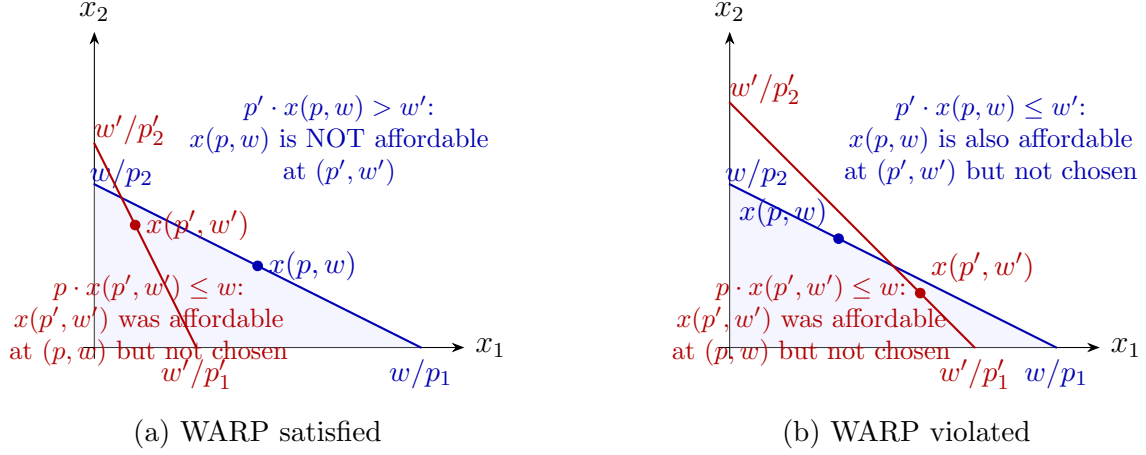


Figure 5: Panel (a): WARP satisfied. $x(p', w')$ (red) lies inside the budget set at (p, w) but was not chosen, revealing $x(p, w) \succsim x(p', w')$. WARP requires $x(p, w)$ to be outside the budget set at (p', w') , i.e. $p' \cdot x(p, w) > w'$, which holds. Panel (b): WARP violated. Both bundles lie inside each other's budget sets: $x(p', w')$ was affordable at (p, w) and $x(p, w)$ was affordable at (p', w') , so the consumer simultaneously reveals $x(p, w) \succsim x(p', w')$ and $x(p', w') \succsim x(p, w)$ with distinct bundles — a contradiction.

affordable and distinct. In the language of §1.C, the demand function $x(\cdot, \cdot)$ satisfies the WARP for the family $\mathcal{B}$ of Walrasian budget sets.

11.2 Relationship to abstract WARP and Sen's α

Abstract WARP. Recall Definition 1.4. A choice correspondence C on a family $\mathcal{B}$ of feasible sets satisfies WARP if, for any $B, B' \in \mathcal{B}$ and $x, y \in B \cap B'$,

$$x \in C(B), \quad y \in C(B') \implies x \in C(B'). \quad (10)$$

This formulation allows multiple selected alternatives. In particular, if x and y are both optimal under both budgets, there is no requirement that $x = y$.

Single-valued choice. Suppose $C(B) = \{c(B)\}$ for every $B \in \mathcal{B}$. Then (10) is equivalent to

$$\boxed{x, y \in B \cap B', \quad x = c(B), \quad y = c(B') \implies x = y.} \quad (11)$$

Indeed, abstract WARP implies $x \in C(B') = \{y\}$, so $x = y$; conversely, (11) implies (10) when C is single-valued. Equivalently, taking the contrapositive,

$$c(B) \neq c(B') \implies c(B') \notin B \text{ or } c(B) \notin B'. \quad (12)$$

Walrasian demand. For $p \gg 0$ and $w > 0$, let $B_{p,w} = \{z \in \mathbb{R}_+^L : p \cdot z \leq w\}$ and let $c(B_{p,w}) = x(p, w)$ be the single-valued demand. Because each demanded bundle is feasible in its own budget, $x(p, w), x(p', w') \in B_{p,w} \cap B_{p',w'}$ holds if and only if

$$p \cdot x(p', w') \leq w \quad \text{and} \quad p' \cdot x(p, w) \leq w'.$$

Thus (11) becomes

$$\boxed{p \cdot x(p', w') \leq w, \quad p' \cdot x(p, w) \leq w' \quad \implies \quad x(p, w) = x(p', w').} \quad (13)$$

Holding the first affordability inequality fixed and taking the contrapositive of the remaining implication yields

$$p \cdot x(p', w') \leq w, \quad x(p', w') \neq x(p, w) \quad \implies \quad p' \cdot x(p, w) > w'. \quad (14)$$

This is exactly Definition 12.1. Hence the abstract WARP, its single-valued formulation, and Definition 12.1 are equivalent for a single-valued choice rule on Walrasian budgets.

Additional comment: Relation to Sen's α . For a single-valued choice function, Sen's contraction axiom is

$$A \subseteq B, \quad c(B) \in A \quad \implies \quad c(A) = c(B) \quad (A, B \in \mathcal{B}). \quad (15)$$

WARP implies α on any domain $\mathcal{B}$: if $A \subseteq B$ and $c(B) \in A$, the two chosen alternatives belong to $A \cap B$, so (11) implies equality. Conversely, if every two-element set belongs to $\mathcal{B}$, then α implies WARP: if distinct $x = c(A)$ and $y = c(B)$ belonged to $A \cap B$, applying α to $\{x, y\}$ would give both $c(\{x, y\}) = x$ and $c(\{x, y\}) = y$.

In particular, when $\mathcal{B}$ contains every nonempty subset of X , single-valued WARP, Sen's α , and rationalizability by a complete and transitive preference with a unique maximizer in every menu are equivalent (the proof of transitivity uses three-element menus). The equivalence with α generally fails on the restricted family of Walrasian budgets: these budgets are not closed under taking arbitrary binary subsets. On that domain WARP still implies α , but α alone need not imply WARP or rationalizability.

11.3 The Compensated Law of Demand

The central testable consequence of WARP is a generalization of the law of demand to compensated price changes.

Definition 11.2 (Slutsky compensation). A price change from p to p' is Slutsky-compensated if wealth is adjusted so that the original demand bundle remains just affordable:

$$w' = p' \cdot x(p, w).$$

Under this compensation, the consumer can still purchase the original bundle $x(p, w)$, so any change in demand reflects pure substitution.

Proposition 11 (Compensated law of demand (MWG Prop. 2.F.1)). *Suppose $x(p, w)$ satisfies Walras' law and WARP. Let p' be any price vector and set $w' = p' \cdot x(p, w)$ (Slutsky compensation). Then*

$$\boxed{(p' - p) \cdot (x(p', w') - x(p, w)) \leq 0.} \quad (16)$$

Moreover, if $x(p', w') \neq x(p, w)$, the inequality is strict.

Economic content of the compensated law of demand

Let $\Delta p = p' - p$ and $\Delta x = x(p', w') - x(p, w)$. The inequality $\Delta p \cdot \Delta x \leq 0$ says: price changes and compensated quantity changes move in opposite directions (on average). In particular, if only the price of good i changes ($p'_j = p_j$ for $j \neq i$) and wealth is Slutsky-compensated, then

$$(p'_i - p_i)(x_i(p', w') - x_i(p, w)) \leq 0,$$

i.e. a rise in p_i leads to a fall (or no change) in compensated demand for good i . This generalises Theorem 3 from Hicksian demand to the WARP setting.

Proof.

Case 1: $x(p', w') = x(p, w)$. Then $\Delta x = 0$ and $(p' - p) \cdot \Delta x = 0 \leq 0$.

Case 2: $x(p', w') \neq x(p, w)$.

Step 1 (Verify WARP applies). By assumption $w' = p' \cdot x(p, w)$, so $x(p, w)$ is affordable at (p', w') :

$$p' \cdot x(p, w) = w'.$$

Since $x(p', w') \neq x(p, w)$, applying WARP (9) with the roles of (p, w) and (p', w') swapped: $p' \cdot x(p, w) = w' \leq w'$ and $x(p, w) \neq x(p', w')$ together imply

$$p \cdot x(p', w') > w.$$

Step 2 (Compute $p' \cdot \Delta x$). By Walras' law $p' \cdot x(p', w') = w'$, and by definition of Slutsky compensation $w' = p' \cdot x(p, w)$. Therefore:

$$p' \cdot (x(p', w') - x(p, w)) = w' - w' = 0.$$

Step 3 (Compute $p \cdot \Delta x$). From Step 1: $p \cdot x(p', w') > w$. By Walras' law: $p \cdot x(p, w) = w$. Therefore:

$$p \cdot (x(p', w') - x(p, w)) = p \cdot x(p', w') - w > 0.$$

Step 4 (Combine).

$$(p' - p) \cdot (x(p', w') - x(p, w)) = \underbrace{p' \cdot \Delta x}_{=0} - \underbrace{p \cdot \Delta x}_{>0} < 0. \quad \square$$

11.4 Implications for the Slutsky Matrix

Definition 11.3. The *Slutsky matrix* $S(p, w)$ is defined as

$$S(p, w) = D_p x(p, w) + D_w x(p, w) \cdot x(p, w)^\top.$$

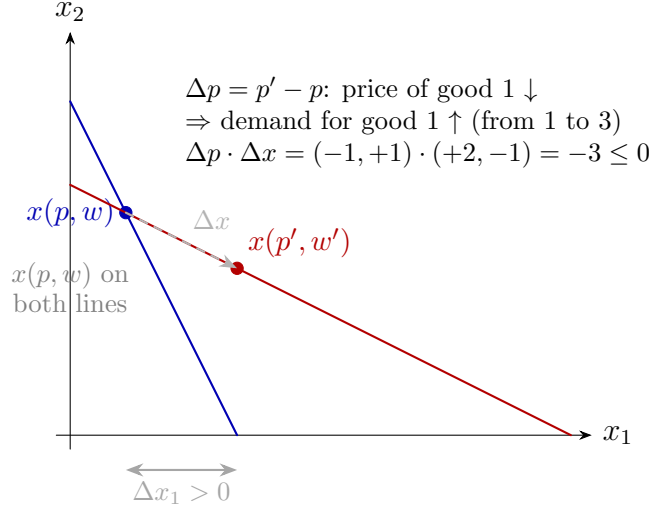


Figure 6: Compensated law of demand. The original budget line (blue line) with (p, w) is steeper (good 1 is expensive relative to good 2). After a Slutsky-compensated price change to (p', w') (good 1 becomes cheaper, $p'_1 < p_1$), the new budget line (red line) with (p, w) passes through $x(p, w)$ and is shallower. Demand shifts from $x(p, w)$ to $x(p', w')$: good 1 demand rises. The inner product $\Delta p \cdot \Delta x < 0$ confirms the compensated law of demand.

Written out explicitly:

$$S(p, w) = \begin{pmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \cdots & \frac{\partial x_1}{\partial p_L} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \cdots & \frac{\partial x_2}{\partial p_L} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_L}{\partial p_1} & \frac{\partial x_L}{\partial p_2} & \cdots & \frac{\partial x_L}{\partial p_L} \end{pmatrix} + \begin{pmatrix} \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial w} \\ \vdots \\ \frac{\partial x_L}{\partial w} \end{pmatrix} (x_1 \quad x_2 \quad \cdots \quad x_L).$$

The generic (i, j) entry is therefore

$$s_{ij}(p, w) = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, w).$$

By the Slutsky equation, this equals the Hicksian cross-price derivative:

$$s_{ij}(p, w) = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j},$$

where $\bar{u} = v(p, w)$ is the indirect utility at (p, w) .

Proposition 12 (WARP implies negative semidefinite Slutsky matrix). *Suppose $x(p, w)$ is differentiable, satisfies Walras' law and WARP. Then the Slutsky substitution matrix $S(p, w)$ is negative semidefinite: $v^\top S(p, w) v \leq 0$ for all $v \in \mathbb{R}^L$.*

Proof sketch. Consider a Slutsky-compensated price change $p' = p + tv$ (for small $t > 0$) with $w' = p' \cdot x(p, w)$.

By first-order Taylor expansion,

$$x(p', w') - x(p, w) \approx [D_p x(p, w) + D_w x(p, w) x(p, w)^\top] tv = S(p, w) tv.$$

For the details see the footnote below.⁵ Substituting into the compensated law of demand (16):

$$(p' - p) \cdot (x(p', w') - x(p, w)) \approx (tv)^\top S(p, w)(tv) = t^2 v^\top S(p, w) v \leq 0.$$

Since $t^2 > 0$, we conclude $v^\top S(p, w) v \leq 0$. □

Remark 23 ($S(p, w)p = 0$). By differentiating Walras' law $p \cdot x(p, w) = w$ with respect to p (holding w fixed) and with respect to w (holding p fixed), one obtains:

$$S(p, w)p = 0.$$

Thus prices always lie in the null space of S ; the matrix S has rank at most $L - 1$.

11.5 WARP versus Full Rationality

WARP and utility maximization (MWG Prop. 2.F.2–2.F.3)

- **Rationality $\Rightarrow$ WARP.** If $x(p, w)$ is derived from utility maximization over a rational preference relation $\succsim$, then $x(p, w)$ satisfies WARP. (This follows because $\succsim$ rationalizes the induced choice rule, which satisfies WARP by Proposition 1 on the choice function.)
- **WARP $\nRightarrow$ Rationality (when $L \geq 3$).** For $L \geq 3$ goods, WARP is strictly weaker than full rationality: there exist demand functions satisfying WARP that cannot be rationalized by any utility function. Intuitively, WARP only restricts

⁵We want to expand $x(p', w')$ around (p, w) where $p' = p + tv$ and $w' = p' \cdot x(p, w) = w + tv \cdot x(p, w)$ (Slutsky compensation). Write $\Delta p = tv$ and $\Delta w = w' - w = tv \cdot x(p, w)$. By differentiability of x in (p, w) :

$$\begin{aligned} x(p', w') &= x(p + tv, w + tv \cdot x(p, w)) \\ &= x(p, w) + D_p x(p, w) \cdot (tv) + D_w x(p, w) \cdot (tv \cdot x(p, w)) + o(t) \\ &= x(p, w) + t \left[D_p x(p, w) \cdot v + D_w x(p, w) (v \cdot x(p, w)) \right] + o(t). \end{aligned}$$

Now note that $v \cdot x(p, w) = x(p, w)^\top v$, so the term in brackets is

$$D_p x(p, w) v + D_w x(p, w) x(p, w)^\top v = \left[D_p x(p, w) + D_w x(p, w) x(p, w)^\top \right] v = S(p, w) v.$$

That's exactly the definition of the Slutsky matrix from Definition 12.3. So:

$$x(p', w') - x(p, w) = t S(p, w) v + o(t).$$

pairwise choices, while rationality requires consistency over all possible budget sets simultaneously.

- **$L = 2$ special case.** When $L = 2$, WARP (together with Walras' law and homogeneity of degree zero) is equivalent to rationalizability: any demand function satisfying these conditions can be derived from a preference relation.
- **Full rationality implies symmetry of S .** Utility maximization implies that $S(p, w)$ is not only negative semidefinite but also symmetric: $S_{ij} = S_{ji}$. WARP alone does not imply symmetry of S when $L \geq 3$; symmetry is the additional restriction imposed by rationality beyond WARP.

Remark 24 (Revealed-preference tests in empirical work). *The compensated law of demand (16) is directly testable from observed demand data: given two price–quantity observations (p, x) and (p', x') , WARP requires that $(p' - p) \cdot (x' - x) \leq 0$ whenever $p \cdot x' \leq p \cdot x$ (i.e. x' was affordable at (p, w)) and $x' \neq x$. More demanding tests based on full rationality are provided by the Strong Axiom of Revealed Preference (SARP) and the Generalised Axiom of Revealed Preference (GARP). See separate note on the relationship among SARP, WARP, and GARP. Different authors use different names. Afriat condition is equivalent with GARP in Echenique and Chambers's book, which seems equivalent to SARP in MWG.*

Remark 25. *The Kihlstrom–Mas-Colell–Sonnenschein (1976) paper (henceforth KMS) provides the definitive characterisation of the set of demand functions satisfying the Weak Axiom of Revealed Preference (WARP). They study the relationship between WARP and NSD of the Slutsky matrix.*

12 Testable implications of utility maximization (MWG §3.E, 3.G)

The demand function $x^*(p, I)$ arises from utility maximization. This origin imposes observable, testable restrictions on the demand function. We have already established two in Section 7.1:

- **Homogeneity of degree zero** (Proposition 6): $x^*(\lambda p, \lambda I) = x^*(p, I)$ for all $\lambda > 0$.
- **Walras' law** (Proposition 7): $p \cdot x^*(p, I) = I$ whenever u is locally nonsatiated.

Two further restrictions emerge from the duality structure of the UMP.

12.1 The Indirect Utility Function and Roy's Identity

Remember that the **indirect utility function** is the value function of the UMP:

$$v(p, I) \equiv \max_{x \geq 0} u(x) \quad \text{subject to} \quad p \cdot x \leq I.$$

It records the maximized utility level attainable at prices $p \in \mathbb{R}_{++}^L$ and income $I > 0$.

The indirect utility function inherits several properties from the structure of the UMP.

Proposition 13 (Properties of $v(p, I)$). *Suppose u is continuous and locally nonsatiated. Then $v(p, I)$ is:*

1. Homogeneous of degree zero in (p, I) : $v(\lambda p, \lambda I) = v(p, I)$ for all $\lambda > 0$.
2. Nondecreasing in I and nonincreasing in each p_i .
3. Quasiconvex in (p, I) .
4. Continuous in (p, I) .

Proof sketch. (i) *Homogeneity.* Multiplying all prices and income by $\lambda > 0$ leaves the budget set $\{x \geq 0 : p \cdot x \leq I\}$ unchanged, so the constrained maximum is unaffected.

(ii) *Monotonicity.* A higher income relaxes the constraint, so the maximum cannot decrease. A higher p_i tightens the constraint, so the maximum cannot increase.

(iii) *Quasiconvexity.* The set of (p, I) that make a given consumption bundle affordable is convex; taking a convex combination of two “budget-constrained” situations leaves the best attainable utility no higher than the better of the two.

(iv) *Continuity.* By the Maximum Theorem: the budget correspondence is continuous in (p, I) and u is continuous, so v is continuous. $\square$

The key link between the indirect utility function and Marshallian demand is **Roy’s identity**, which allows one to recover the entire demand system from the single scalar function v .

Proposition 14 (Roy’s Identity (MWG Prop. 3.G.4)). *Suppose $v(p, I)$ is differentiable and $\frac{\partial v}{\partial I}(p, I) \neq 0$. Then the Marshallian demand for good i satisfies*

$$x_i^*(p, I) = - \frac{\partial v(p, I) / \partial p_i}{\partial v(p, I) / \partial I}.$$

Proof. The Lagrangian for the UMP is $\mathcal{L} = u(x) - \lambda(p \cdot x - I)$, with optimal Lagrange multiplier $\lambda^* \geq 0$ (the marginal utility of income). By the **envelope theorem**, the partial derivatives of the value function v with respect to parameters equal the corresponding partial derivatives of $\mathcal{L}$ evaluated at the optimum:

$$\frac{\partial v}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} \Big|_{x=x^*} = -\lambda^* x_i^*, \quad \frac{\partial v}{\partial I} = \frac{\partial \mathcal{L}}{\partial I} \Big|_{x=x^*} = \lambda^*.$$

Dividing gives $\partial v / \partial p_i / \partial v / \partial I = -x_i^*$, which rearranges to Roy’s identity. $\square$

Roy’s Identity: recovering demand from the indirect utility function

Given the indirect utility function $v(p, I)$, Marshallian demands are recovered by:

$$x_i^*(p, I) = - \frac{\partial v(p, I) / \partial p_i}{\partial v(p, I) / \partial I}, \quad i = 1, \dots, L.$$

Intuition: A one-unit rise in p_i reduces utility by $\lambda^* x_i^*$ (it costs x_i^* extra units of income, each worth λ^* in utility). Dividing by λ^* converts this utility loss back into income units, yielding x_i^* . Thus the quantity demanded equals the income-equivalent utility cost of a unit price rise.

Example 9 (Roy's identity for Cobb-Douglas). For $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ with $\alpha \in (0, 1)$, the indirect utility function (substituting the optimal demands $x_1^* = \alpha I/p_1$, $x_2^* = (1 - \alpha)I/p_2$) is

$$v(p_1, p_2, I) = \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} I.$$

Applying Roy's identity to good 1:

$$x_1^* = -\frac{\partial v / \partial p_1}{\partial v / \partial I} = -\frac{-\alpha v / p_1}{v / I} = \frac{\alpha I}{p_1},$$

recovering the Cobb-Douglas Marshallian demand.

12.2 Symmetry of the Slutsky Matrix

We showed in Section 11 that WARP implies the Slutsky matrix is negative semidefinite (Proposition 12). Full utility maximization—rationality with complete and transitive preferences—implies the stronger result that the Slutsky matrix is also symmetric.

Recall from the Slutsky equation (Theorem 4) that the (i, j) entry of the Slutsky matrix is

$$s_{ij}(p, I) = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} = \frac{\partial x_i(p, I)}{\partial p_j} + \frac{\partial x_i(p, I)}{\partial I} x_j^*(p, I).$$

Proposition 15 (Slutsky symmetry (MWG Prop. 3.G.1)). If $x^*(p, I)$ is derived from utility maximization by a continuous, locally nonsatiated utility function, then the Slutsky matrix $S(p, I)$ is symmetric: $s_{ij} = s_{ji}$ for all i, j .

Proof. By Shephard's Lemma (Proposition 9), Hicksian demand is the gradient of the expenditure function:

$$x_i^h(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_i}.$$

Therefore

$$s_{ij} = \frac{\partial x_i^h}{\partial p_j} = \frac{\partial^2 e}{\partial p_j \partial p_i}.$$

By **Young's theorem** (equality of mixed partial derivatives), since e is twice continuously differentiable:

$$\frac{\partial^2 e}{\partial p_j \partial p_i} = \frac{\partial^2 e}{\partial p_i \partial p_j} = s_{ji}.$$

Hence S is symmetric. □

Slutsky symmetry: a cross-equation restriction on demand

Utility maximization imposes, for all pairs $i \neq j$:

$$\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial I} x_j^* = \frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial I} x_i^*.$$

This is a testable restriction on the Marshallian demand system. It follows purely from the fact that both x_i^h and x_j^h are partial derivatives of the same scalar function $e(p, \bar{u})$, so they must satisfy the cross-derivative condition (Young's theorem). With L goods there are $\frac{L(L-1)}{2}$ such restrictions.

Remark 26 (WARP vs. full rationality). *For $L = 2$ goods, NSD of the Slutsky matrix already forces symmetry (a 2×2 matrix satisfying $p^\top S = 0$ and $S \preceq 0$ must be symmetric). For $L \geq 3$ goods, symmetry is an additional restriction beyond WARP: the Slutsky matrix can be NSD but not symmetric. The Strong Axiom of Revealed Preference (SARP), which is equivalent to full rationality, is needed to obtain symmetry for $L \geq 3$.*

12.3 The Integrability Theorem

The four properties established above—homogeneity, Walras' law, Slutsky symmetry, and Slutsky NSD—are not only necessary for utility maximization: they are also sufficient.

Theorem 5 (Integrability Theorem (MWG Prop. 3.H.1)). *A continuously differentiable demand function $x(p, I)$ on $\mathbb{R}_{++}^L \times \mathbb{R}_{++}$ is rationalised by a continuous, locally nonsatiated utility function if and only if it satisfies:*

1. **Homogeneity of degree zero:** $x(\lambda p, \lambda I) = x(p, I)$ for all $\lambda > 0$.
2. **Walras' law:** $p \cdot x(p, I) = I$ for all (p, I) .
3. **Slutsky symmetry:** The Slutsky matrix $S(p, I)$ is symmetric.
4. **Slutsky NSD:** $z^\top S(p, I) z \leq 0$ for all $z \in \mathbb{R}^L$.

Proof sketch. **Necessity** (utility maximization $\Rightarrow$ all four conditions):

- (i) and (ii): Propositions 6 and 7.
- (iii) Symmetry: Proposition 15 via Young's theorem.
- (iv) NSD: By the compensated law of demand (Proposition 11 and 12), $dp \cdot dx^h \leq 0$ for any compensated price change, i.e. $dp^\top S dp \leq 0$ for all dp .

Sufficiency (all four conditions $\Rightarrow$ rationalisability): Given $x(p, I)$ satisfying (i)–(iv), construct the expenditure function by integrating:

$$e(p, \bar{u}) = \int_{p^0}^p x^h(q, \bar{u}) \cdot dq,$$

where x^h is defined implicitly via the Slutsky equation. This is a line integral of a vector field. See appendix for the detail.

- Symmetry (iii) is the integrability condition. By the Frobenius theorem, the PDE system $\partial e / \partial p_i = x_i^h(p)$ has a (path-independent) solution if and only if $\partial h_i / \partial p_j = \partial h_j / \partial p_i$ —which is exactly Slutsky symmetry. Hence e is well-defined. See the remark below.
- NSD (iv) ensures concavity. Since $\partial^2 e / \partial p_i \partial p_j = s_{ij}$ and $S \preceq 0$, the constructed e is concave in p , as required for a valid expenditure function.
- The utility function is recovered via the duality $u(x) = \min_p \{v : e(p, v) \geq p \cdot x\}$ by duality. This says: the utility of bundle x is the highest utility level that x can “afford” at every price vector simultaneously.

□

Remark 27. (*The Construction of u : Support Function Duality*) The recovery formula rests on the **supporting hyperplane theorem** for closed convex sets.

Step 1: Upper contour sets and the expenditure function. For each utility level v , the upper contour set is

$$U(v) = \{x \in \mathbb{R}_+^N : u(x) \geq v\},$$

which is convex (by quasi-concavity of u) and closed (by continuity of u). The expenditure function, for fixed v , is precisely the support function of $U(v)$:

$$e(p, v) = \min_{x \in U(v)} p \cdot x.$$

Geometrically, $e(p, v)$ is the signed distance (scaled by $\|p\|$) from the origin to the supporting hyperplane of $U(v)$ with outward normal p .

Step 2: Recovering $U(v)$ via the supporting hyperplane theorem. By the supporting hyperplane theorem, any closed convex set equals the intersection of all half-spaces that contain it. Applied to $U(v)$:

$$U(v) = \{x \in \mathbb{R}_+^N : p \cdot x \geq e(p, v) \ \forall p \gg 0\}.$$

Each price vector p defines a half-space $\{x : p \cdot x \geq e(p, v)\}$, and $U(v)$ is exactly their intersection.

Step 3: Recovering $u(x)$. From Step 2, $x \in U(v)$ —equivalently, $u(x) \geq v$ —if and only if $p \cdot x \geq e(p, v)$ for all $p \gg 0$. Taking the supremum over all such v yields:

$$u(x) = \sup \{v : e(p, v) \leq p \cdot x \ \forall p \gg 0\}.$$

Since e is continuous and strictly increasing in v (by local nonsatiation), the supremum is attained and we may write $\max$ in place of $\sup$.

The four testable conditions for rationalisability

A demand system $x(p, I)$ is consistent with utility maximization if and only if:

- | | | |
|-------|------------------|---|
| (i) | Homogeneity | $x(\lambda p, \lambda I) = x(p, I)$ for all $\lambda > 0$ |
| (ii) | Walras' law | $p \cdot x(p, I) = I$ |
| (iii) | Slutsky symmetry | $s_{ij} = s_{ji}$ for all $i \neq j$ |
| (iv) | Slutsky NSD | $z^\top S(p, I) z \leq 0$ for all $z \in \mathbb{R}^L$ |

Conditions (i) and (ii) are linear restrictions, easy to impose by construction. Condition (iii) provides $\frac{L(L-1)}{2}$ cross-equation restrictions and is the key integrability condition (it makes the expenditure function well-defined). Condition (iv) adds the curvature requirement. Together they are the complete set of restrictions imposed by consumer rationality.

Remark 28 (Empirical testing of rationality). *In applied demand analysis one estimates a parametric demand system (e.g. the Almost Ideal Demand System) and tests whether the estimated Slutsky matrix is symmetric and NSD. Rejection of symmetry is the standard test of the rational-consumer hypothesis. Rejection of NSD alone (without symmetry failure) is far less common.*

Remark 29 (Summary: the duality cycle). *The integrability theorem closes the loop of consumer duality:*

$$\underbrace{u(x)}_{\text{direct utility}} \xrightarrow{\text{UMP}} \underbrace{x^*(p, I)}_{\text{Marshallian}} \xrightarrow{\text{Slutsky}} \underbrace{x^h(p, \bar{u})}_{\text{Hicksian}} = \nabla_p \underbrace{e(p, \bar{u})}_{\text{expenditure fn.}} \xrightarrow{\text{duality}} u(x).$$

The Integrability Theorem says: any x^* satisfying conditions (i)–(iv) is a valid entry point into this cycle and comes from some well-behaved utility function.

13 Welfare Evaluation of Price Changes

In this section we study how to measure, in monetary terms, the welfare gain or loss a consumer experiences when prices change. We develop three measures—the compensating variation (CV), the equivalent variation (EV), and the area variation (AV, also called the change in Marshallian consumer surplus)—and establish their relationship.

Throughout, we consider a consumer with a continuous, strictly quasi-concave, and locally nonsatiated utility function $u : \mathbb{R}_+^N \rightarrow \mathbb{R}$. The consumer faces an initial price vector p^0 and a new price vector p^1 , with income I held fixed. For notational simplicity, we focus on a change in the price of good 1 alone: $p_1^0 = a$ and $p_1^1 = b$ with $b > a$ (a price increase), while $p_{-1}^0 = p_{-1}^1 = \bar{p}_{-1}$.

Example 10 (Electricity price increase). *An earthquake damages a power plant, forcing an electricity company to raise the price of power from a to b . The government orders the company to compensate consumers. How much compensation is required to leave consumers no worse off than before?*

Denote by $u^0 = v(p^0, I)$ and $u^1 = v(p^1, I)$ the utility levels before and after the price change, where v is the indirect utility function. Then $u^1 < u^0$.

13.1 A Money Metric: The Indirect Utility Function

Given any price vector p and any indirect utility function $v(p, I)$, we know that v is strictly increasing in I (by local nonsatiation). Hence, for a fixed reference price vector $\bar{p}$, the function

$$e(\bar{p}, v(p, I))$$

is a money metric indirect utility function: it measures utility in dollar units at reference prices $\bar{p}$. It is strictly increasing in I and provides a valid indirect utility function representing $\succsim$.

If $v(p', I) > v(p^0, I)$, then $e(\bar{p}, v(p', I)) > e(\bar{p}, v(p^0, I))$: the consumer is better off, and the difference is expressed in dollars. Both CV and EV are instances of this idea, differing only in the choice of reference prices.

13.2 Compensating Variation

Definition 13.1 (Compensating Variation). The **compensating variation** (CV) is the amount of income the consumer must receive after the price change to restore the original utility level u^0 :

$$\text{CV} \equiv e(p^1, u^0) - e(p^0, u^0).$$

Since $e(p^0, u^0) = I$, this simplifies to

$$\text{CV} = e(p^1, u^0) - I.$$

Interpretation. The CV answers: “After the price change occurs, how much additional income would the consumer need to be just as well off as before?” Equivalently, $CV = e(p^1, u^0) - e(p^0, u^0)$, which is the difference between the cost of achieving u^0 and the cost of achieving u^1 at the new prices—both evaluated at p^1 .

For a price increase ($b > a$), the consumer is worse off, so $CV > 0$: the consumer needs a positive transfer to be compensated.

CV as area under compensated demand. By Shephard’s lemma, $\partial e(p, u^0)/\partial p_1 = x_1^h(p, u^0)$, so:

$$CV = e(p^1, u^0) - e(p^0, u^0) = \int_a^b x_1^h(p_1, \bar{p}_{-1}^0, u^0) dp_1, \quad (17)$$

where $\bar{p}_{-1}^0$ is the fixed price of the other goods (other than good 1), $p^0 = (p_1^0, \bar{p}_{-1}^0) = (a, \bar{p}_{-1}^0)$ and $p^1 = (p_1^1, \bar{p}_{-1}^0) = (b, \bar{p}_{-1}^0)$. The CV is the area to the left of the initial Hicksian (compensated) demand curve $x_1^h(\cdot, u^0)$ between $p_1^0 = a$ and $p_1^1 = b$.

13.3 Equivalent Variation

Definition 13.2 (Equivalent Variation). The **equivalent variation** (EV) is the amount of income the consumer would be willing to pay, before the price change, to avoid it:

$$EV \equiv e(p^1, u^1) - e(p^0, u^1).$$

Since $e(p^1, u^1) = I$, this simplifies to

$$EV = I - e(p^0, u^1).$$

Interpretation. The EV answers: “At the old prices, how much income would the consumer give up to prevent the price change?” Equivalently, EV is the income-equivalent of the welfare change measured at the old prices.

For a price increase, $EV > 0$: the consumer would pay a positive amount to avoid it.

EV as area under compensated demand. By Shephard’s lemma applied at utility level u^1 :

$$EV = e(p^1, u^1) - e(p^0, u^1) = \int_a^b x_1^h(p_1, \bar{p}_{-1}, u^1) dp_1. \quad (18)$$

The EV is the area to the left of the final Hicksian demand curve $x_1^h(\cdot, u^1)$ between $p_1^0 = a$ and $p_1^1 = b$.

13.4 The Area Variation (Marshallian Consumer Surplus)

A practical difficulty with both CV and EV is that the Hicksian demand functions $x_1^h(\cdot, u^0)$ and $x_1^h(\cdot, u^1)$ are not directly observable. Only the Marshallian demand $x_1(p, I)$ can be estimated from data. This motivates a third measure.

Definition 13.3 (Area Variation). The **area variation** (AV), also called the change in **Marshallian consumer surplus**, is

$$AV = \int_a^b x_1(p_1, \bar{p}_{-1}, I) dp_1. \quad (19)$$

It is the area to the left of the Marshallian (ordinary) demand curve between $p_1 = a$ and $p_1 = b$.

The AV is directly computable from observable demand data and does not require knowledge of the consumer's preferences.

Remark 30. *AV is often referred to as consumer surplus, largely for historical reasons tracing back to Marshall's terminology. Subsequent work questioned the welfare interpretation of this measure and introduced equivalent variation (EV) and compensating variation (CV) as theoretically grounded alternatives. AV is best viewed as a practical approximation to EV and CV—an interpretation formalized by Willig.*

AV may not be uniquely defined. See the appendix.

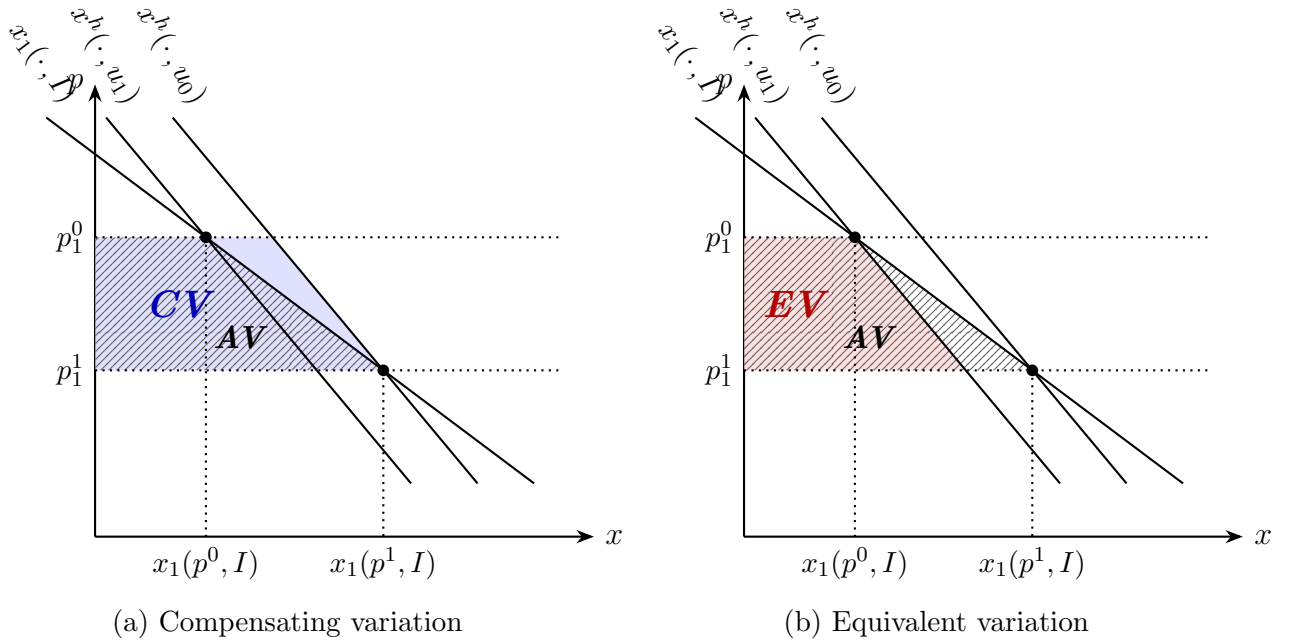


Figure 7: EV and CV from Hicksian demand curves. Price increases from p_1^0 to p_1^1 , decreasing utility from u_0 to u_1 . The two steeper lines are the Hicksian demands $x^h(\cdot, u_0)$ and $x^h(\cdot, u_1)$; the less steep line is the Marshallian demand $x_1(\cdot, I)$ as we are assuming normal goods. Panel (a): EV (blue) is the area to the left of $x^h(\cdot, u_1)$; AV (dashed) is the area to the left of $x_1(\cdot, I)$. Panel (b): CV (red) is the area to the left of $x^h(\cdot, u_0)$; AV (dashed) is the area to the left of $x_1(\cdot, I)$. For a normal good with a price increase, $CV > AV > EV$.

13.5 The Ordering: CV, AV, and EV

The three measures are related through the Slutsky equation. The key observation is that, for a price increase of a normal good, the Hicksian demand curve at utility u^0 lies to the right of the Marshallian demand curve, which in turn lies to the right of the Hicksian demand curve at u^1 .

Proposition 16 (Ordering of welfare measures for normal goods). *Suppose good 1 is a normal good ($\partial x_1 / \partial I > 0$) and $b > a$ (a price increase). Then:*

$$CV \geq AV \geq EV > 0.$$

Proof. The Slutsky equation gives, for each p_1 ,

$$\frac{\partial x_1^h(p_1, u)}{\partial p_1} = \frac{\partial x_1(p_1, I)}{\partial p_1} + \frac{\partial x_1(p_1, I)}{\partial I} x_1.$$

Since the income effect $(\partial x_1 / \partial I) x_1 > 0$ for a normal good, the Hicksian demand curve is steeper (less elastic) than the Marshallian demand curve. Moreover, since $u^0 > u^1$ (the price increase lowers utility) and good 1 is normal, we have

$$x_1^h(p_1, u^0) \geq x_1(p_1, I) \geq x_1^h(p_1, u^1) \quad \text{for all } p_1 \in [a, b].$$

Integrating from a to b yields $CV \geq AV \geq EV$. □

Remark 31 (When all three coincide). *If the utility function is quasi-linear in a numeraire good, i.e. $u(x_1, \dots, x_{N-1}, x_N) = \phi(x_1, \dots, x_{N-1}) + x_N$, then the income effect for goods $1, \dots, N-1$ is identically zero. In this case, the Hicksian and Marshallian demand curves coincide, and*

$$CV = AV = EV.$$

More generally, when the income effect is small (small budget share $w_1 = p_1 x_1 / I$), all three measures are approximately equal, and AV is a good approximation to both CV and EV .

Appendix

A Integrability Condition and Slutsky symmetry

The question is as follows:

Does there exist a scalar function $e : \mathbb{R}_{++}^L \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}) \quad \text{for all } i = 1, \dots, L?$$

In the language of multivariable calculus, this asks: is the vector field $h = (h_1, \dots, h_L)$ the gradient of some scalar function? Equivalently: is the differential 1-form

$$\omega = \sum_{i=1}^L x_i^h(p, \bar{u}) dp_i$$

exact?

A.1 Math Review: Exact Differential Forms and Conservative Vector Fields

Recall from multivariable calculus the following classical result.

Theorem 6 (Poincaré Lemma / Exactness Criterion). *Let $\Omega \subseteq \mathbb{R}^L$ be an open, simply connected domain, and let $f_1, \dots, f_L : \Omega \rightarrow \mathbb{R}$ be C^1 functions. The following are equivalent:*

(a) **Exactness.** *There exists a C^2 function $F : \Omega \rightarrow \mathbb{R}$ such that*

$$\frac{\partial F}{\partial p_i} = f_i(p) \quad \text{for all } i.$$

(b) **Path independence.** *For any two points $p^0, p^1 \in \Omega$ and any piecewise- C^1 path γ from p^0 to p^1 ,*

$$\int_{\gamma} \sum_i f_i(p) dp_i$$

depends only on the endpoints, not on the path γ .

(c) **Symmetry of the Jacobian.** *For all $i, j \in \{1, \dots, L\}$,*

$$\frac{\partial f_i}{\partial p_j} = \frac{\partial f_j}{\partial p_i}.$$

Remark 32. *In $L = 2$ dimensions, condition (c) reduces to the familiar $\partial f_1 / \partial p_2 = \partial f_2 / \partial p_1$, which is the criterion for an exact differential $f_1 dp_1 + f_2 dp_2 = dF$. In $L = 3$, it is equivalent to $\nabla \times f = 0$ (the vector field is irrotational / conservative). In general L dimensions, the result is a special case of the Frobenius theorem (or equivalently, the Poincaré lemma for 1-forms on simply connected domains).*

The logical structure is:

(a) Exactness $\implies$ (c) Symmetry

is immediate from Young's theorem ($\partial^2 F / \partial p_j \partial p_i = \partial^2 F / \partial p_i \partial p_j$). The substantive content is the converse:

(c) Symmetry $\implies$ (a) Exactness,

which says that the Jacobian symmetry condition is sufficient for a potential function to exist.

Application to Consumer Theory

Now set $f_i(p) = x_i^h(p, \bar{u})$ (Hicksian demand for good i , with $\bar{u}$ fixed) and $F(p) = e(p, \bar{u})$ (the expenditure function we wish to construct). The domain $\Omega = \mathbb{R}_{++}^L$ is open and simply connected (it is convex). The theorem then says:

The expenditure function $e(p, \bar{u})$ satisfying $\nabla_p e = h(\cdot, \bar{u})$ exists $\iff$ the Jacobian of h is symmetric:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_j} = \frac{\partial h_j(p, \bar{u})}{\partial p_i} \quad \text{for all } i, j.$$

But $\partial x_i^h / \partial p_j = s_{ij}$ is exactly the (i, j) entry of the Slutsky matrix. So the symmetry condition is:

$$s_{ij} = s_{ji} \quad \text{for all } i, j,$$

which is **Slutsky symmetry**.

A.2 Constructing e by Integration

When symmetry holds, the expenditure function is recovered by a line integral. Fix a reference price p^0 and a reference utility level $\bar{u}$ with $e(p^0, \bar{u})$ known (e.g. from observed data). For any target price p , choose any piecewise- C^1 path γ from p^0 to p in $\mathbb{R}_{++}^L$ and define:

$$e(p, \bar{u}) = e(p^0, \bar{u}) + \int_{\gamma} \sum_{i=1}^L x_i^h(q, \bar{u}) dq_i.$$

Slutsky symmetry guarantees this integral is **path-independent**, so e is well-defined regardless of which path γ is chosen.

A.3 Why Symmetry Guarantees Path Independence: A Direct Argument

We verify directly that the Jacobian symmetry condition implies path independence. Consider two paths γ_1 and γ_2 from p^0 to p , and let $\Gamma = \gamma_1 - \gamma_2$ denote the closed loop traversing γ_1 forward and γ_2 backward. We need to show:

$$\oint_{\Gamma} \sum_i x_i^h dq_i = 0.$$

By Stokes' theorem (Green's theorem in $L = 2$), this loop integral equals a surface integral of the “curl” of h . In two dimensions:

$$\oint_{\Gamma} h_1 dq_1 + h_2 dq_2 = \iint_D \left(\frac{\partial h_2}{\partial q_1} - \frac{\partial h_1}{\partial q_2} \right) dq_1 dq_2,$$

where D is the region enclosed by Γ . If Slutsky symmetry holds ($\partial h_1 / \partial q_2 = \partial h_2 / \partial q_1$), the integrand vanishes identically, so the loop integral is zero. The argument extends to

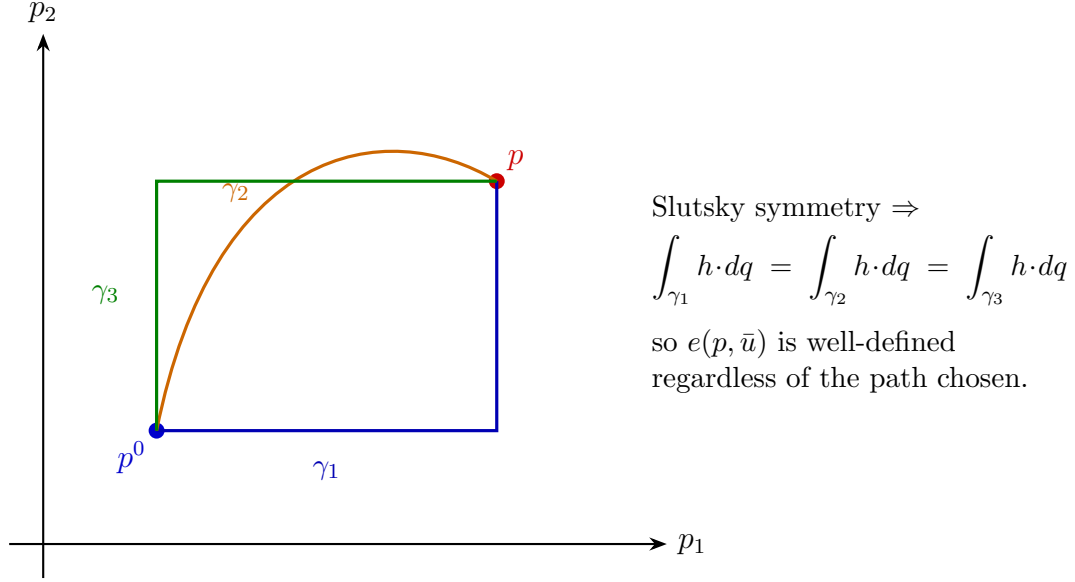


Figure 8: Path independence of the line integral. Slutsky symmetry ensures that integrating the Hicksian demand vector field $h = (h_1, \dots, h_L)$ from p^0 to p gives the same value along any path γ in $\mathbb{R}_{++}^L$.

$L \geq 3$ dimensions via the general Stokes' theorem: the exterior derivative of the 1-form $\omega = \sum_i x_i^h dp_i$ is

$$d\omega = \sum_{i < j} \left(\frac{\partial h_j}{\partial p_i} - \frac{\partial x_i^h}{\partial p_j} \right) dp_i \wedge dp_j.$$

Slutsky symmetry ($s_{ij} = s_{ji}$) makes every coefficient zero, so $d\omega = 0$, which means ω is closed. On the simply connected domain $\mathbb{R}_{++}^L$, closed $\Leftrightarrow$ exact (by the Poincaré lemma), so $\omega = de$ for some function e .

A.4 What Goes Wrong Without Symmetry

If Slutsky symmetry fails—say $s_{12} \neq s_{21}$ —then different paths of integration from p^0 to p yield different values:

$$\int_{\gamma_1} h \cdot dq \neq \int_{\gamma_2} h \cdot dq.$$

No single-valued function $e(p, \bar{u})$ can simultaneously satisfy $\nabla_p e = h$, because a gradient field must have a symmetric Jacobian (Young's theorem). In economic terms: the demand system cannot have arisen from any well-behaved utility function, because the expenditure function that would rationalise it simply does not exist.

A.5 Concrete Example: $L = 2$

Example 11. Consider a candidate Hicksian demand system:

$$h_1(p_1, p_2, \bar{u}) = \frac{\bar{u}}{p_1^{1/2} p_2^{1/2}}, \quad h_2(p_1, p_2, \bar{u}) = \frac{\bar{u}}{p_1^{1/2} p_2^{1/2}}.$$

Check symmetry:

$$\frac{\partial h_1}{\partial p_2} = -\frac{\bar{u}}{2 p_1^{1/2} p_2^{3/2}}, \quad \frac{\partial h_2}{\partial p_1} = -\frac{\bar{u}}{2 p_1^{3/2} p_2^{1/2}}.$$

These are not equal (unless $p_1 = p_2$), so $s_{12} \neq s_{21}$. No expenditure function exists—this demand system is **not rationalisable**.

Now consider the Cobb–Douglas Hicksian demands (with equal shares):

$$h_1(p_1, p_2, \bar{u}) = \bar{u} \left(\frac{p_2}{p_1} \right)^{1/2}, \quad h_2(p_1, p_2, \bar{u}) = \bar{u} \left(\frac{p_1}{p_2} \right)^{1/2}.$$

Check symmetry:

$$\frac{\partial h_1}{\partial p_2} = \frac{\bar{u}}{2 p_1^{1/2} p_2^{1/2}}, \quad \frac{\partial h_2}{\partial p_1} = \frac{\bar{u}}{2 p_1^{1/2} p_2^{1/2}}.$$

These are equal, so $s_{12} = s_{21}$. The expenditure function exists and equals $e(p, \bar{u}) = 2\bar{u} p_1^{1/2} p_2^{1/2}$, which one can verify by checking $\partial e / \partial p_i = x_i^h$.

B Path-Dependence of Consumer Surplus with Multiple Price Change

B.1 The Problem: Multi-Price Changes and Path-Dependence

When only a single price changes (say p_1 moves from a to b , all else fixed), the Marshallian consumer surplus

$$\Delta S = - \int_a^b x_1(p_1, \bar{p}_{-1}, I) dp_1$$

is an ordinary one-dimensional integral and is unambiguous.

Auerbach’s two-market illustration. Let x^i be demand for good i . Consider two goods. If we change p^1 first and then p^2 , the surplus change is

$$\Delta S_1 = - \int_{p_0^1}^{p_1^1} x^1(p^1, p_0^2) dp^1 - \int_{p_0^2}^{p_1^2} x^2(p_1^1, p^2) dp^2.$$

If instead we change p^2 first and then p^1 :

$$\Delta S_2 = - \int_{p_0^1}^{p_1^1} x^1(p^1, p_1^2) dp^1 - \int_{p_0^2}^{p_1^2} x^2(p_0^1, p^2) dp^2.$$

The difference is

$$\Delta S_2 - \Delta S_1 = - \int_{p_0^1}^{p_1^1} [x^1(p^1, p_1^2) - x^1(p^1, p_0^2)] dp^1 + \int_{p_0^2}^{p_1^2} [x^2(p_1^1, p^2) - x^2(p_0^1, p^2)] dp^2.$$

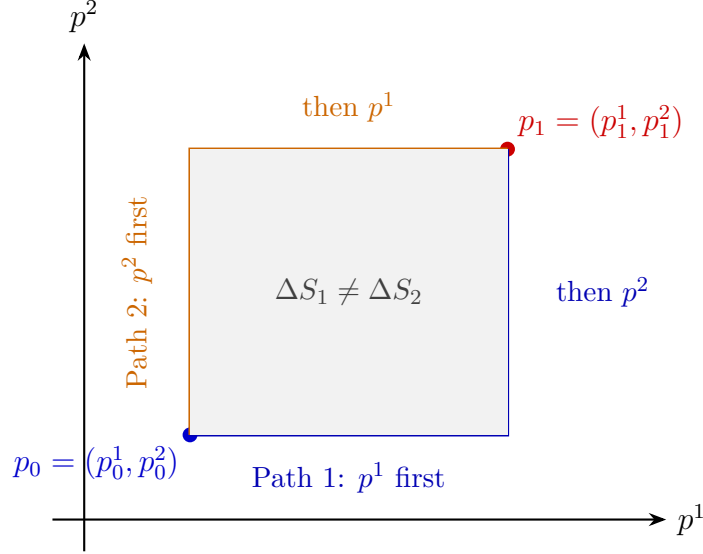


Figure 9: Path-dependence of Marshallian consumer surplus. The line integral $\Delta S = -\int \sum x^i dp^i$ from p_0 to p_1 depends on whether we change p^1 first (Path 1) or p^2 first (Path 2). The two values differ whenever $\partial x^1/\partial p^2 \neq \partial x^2/\partial p^1$.

For infinitesimal changes ($p_1^1 = p_0^1 + dp^1$, $p_1^2 = p_0^2 + dp^2$), this reduces to

$$\Delta S_2 - \Delta S_1 = \left(\frac{\partial x^2}{\partial p^1} - \frac{\partial x^1}{\partial p^2} \right) dp^1 dp^2,$$

which vanishes if and only if $\partial x^1/\partial p^2 = \partial x^2/\partial p^1$ —the exactness condition for the 1-form $\omega = -\sum_i x^i dp^i$.

B.2 Subsequent Literature

Several important developments followed Auerbach's survey, addressing the question of when and how well Marshallian surplus approximates the true welfare change.

Willig (1976). Showed that for a single price change, the percentage error from using Marshallian surplus in place of CV or EV is bounded by a simple function of the income elasticity η and the surplus-to-income ratio $\Delta S/I$. Specifically:

$$\frac{|\text{CV} - \Delta S|}{|\Delta S|} \leq \frac{|\eta|}{2} \frac{|\Delta S|}{I}.$$

For goods with small budget shares and moderate income elasticities, this error is negligible. This result was highly influential in applied work, as it justified using Marshallian surplus for single-market welfare calculations.

Hausman (1981). Rather than approximating, Hausman showed how to recover the exact Hicksian demand (and hence exact CV) from an estimated Marshallian demand system. The

method integrates the expenditure function using Roy's identity and the Slutsky equation. Hausman applied this to U.S. labour supply to show that the deadweight loss from income taxation was substantially larger than the Harberger triangle approximation suggested. The "exact consumer surplus" approach became standard in applied industrial organisation and public economics.

Chipman and Moore (1976, 1980). Studied the multi-price-change problem directly and proved a sharp impossibility result:

Proposition 17 (Chipman–Moore). *Consumer surplus provides a welfare ordering consistent with utility across all possible multi-price changes if and only if preferences admit a quasi-linear representation (zero income effects for all priced goods).*

This means there is no way to rescue consumer surplus as a general multi-price welfare measure without giving up income effects entirely. The result underscores that CV and EV are the only theoretically sound measures for general welfare evaluation.

McKenzie and Pearce (1976). Independently established that Marshallian surplus is path-independent if and only if the Marshallian cross-price Jacobian $\partial x^i / \partial p^j$ is symmetric—which, by the Slutsky equation, requires zero income effects (or the degenerate case of a single good).

Vives (1987). Extended the analysis to show that consumer surplus remains a valid local approximation for small price changes, even with multiple prices. The path-dependence error is second-order in the size of the price change: the Jacobian asymmetry $(\partial x^i / \partial p^j - \partial x^j / \partial p^i)$ is multiplied by the product $dp^i dp^j$. Thus, for small simultaneous price changes, the three measures CV, AV, and EV all agree to first order, and the path of integration becomes irrelevant to that order of approximation.

B.3 Implications for Applied Welfare Analysis

Scenario	Recommended measure	Justification
Single price change, small budget share	AV (Marshallian surplus)	Willig bounds: error is negligible
Single price change, exact welfare	CV or EV via Hausman method	Exact; no approximation error
Multiple price changes	CV or EV only	AV is path-dependent (Chipman–Moore)
Multiple small price changes	AV acceptable as approximation	Path-dependence is second-order (Vives)
Quasi-linear preferences	$AV = CV = EV$	All three coincide; no path-dependence

The overarching conclusion: modern applied welfare economics predominantly uses CV and EV rather than Marshallian surplus, especially when multiple prices change or when precision matters. The Marshallian surplus remains a useful shortcut for single-market, small-change analysis, but its theoretical foundations are limited.

References

- Auerbach, A. J. (1985). “The Theory of Excess Burden and Optimal Taxation.” *Handbook of Public Economics*, Vol. 1, Ch. 2.
- Chipman, J. S. and J. C. Moore (1980). “Compensating Variation, Consumer’s Surplus, and Welfare.” *American Economic Review*, 70(5), 933–949.
- Hausman, J. A. (1981). “Exact Consumer’s Surplus and Deadweight Loss.” *American Economic Review*, 71(4), 662–676.
- McKenzie, G. W. and I. F. Pearce (1976). “Exact Measures of Welfare and the Cost of Living.” *Review of Economic Studies*, 43(3), 465–468.
- Vives, X. (1987). “Small Income Effects: A Marshallian Theory of Consumer Surplus and Downward Sloping Demand.” *Review of Economic Studies*, 54(1), 87–103.
- Willig, R. D. (1976). “Consumer’s Surplus Without Apology.” *American Economic Review*, 66(4), 589–597.