

Consumption Theory: Problem Set 1

Preferences, choice rules, and optimal consumption

Student version: questions only

Instructions. Answer all four questions and justify your answers. Each question is worth 25 points; bracketed numbers give the points for each part. The total is 100 points.

Reading. Saito, *Lecture Note on Consumption Theory* (September 15, 2026), Sections 1–7. You may use elementary calculus and the definitions in the handout. The general programming theorem from the old homework is not required.

1. Constructing and normalizing a utility function

25 points

Let $X = \mathbb{R}_+^N$, and let $\succeq$ be complete, transitive, continuous, and monotonic in the handout's sense: $x \geq y$ coordinatewise and $x \neq y$ imply $x \succ y$. Write $\mathbf{1} = (1, \dots, 1)$. For each $x \in X$, define

$$B(x) = \{t \geq 0 : t\mathbf{1} \succeq x\}, \quad W(x) = \{t \geq 0 : x \succeq t\mathbf{1}\}.$$

Do not invoke a utility-representation theorem; construct the representation directly.

- (a) **[5]** Define $m(x) = \min_i x_i$ and $M(x) = \max_i x_i$. Prove that $m(x) \in W(x)$, $M(x) \in B(x)$, and $B(x) \cup W(x) = \mathbb{R}_+$.
- (b) **[6]** Prove that $B(x)$ and $W(x)$ are closed subsets of $\mathbb{R}_+$. Use connectedness to show that $B(x) \cap W(x) \neq \emptyset$. *Hint:* Consider the map $d(t) = t\mathbf{1}$ and inverse images of contour sets.
- (c) **[8]** Prove that the intersection contains a unique number α_x . Define $u(x) = \alpha_x$. Show that u represents $\succeq$, is monotonic, and satisfies

$$u(t\mathbf{1}) = t, \quad m(x) \leq u(x) \leq M(x).$$

- (d) **[6]** Prove that u is continuous on X , including at boundary points. *Hint:* For $c \geq 0$, express the sets $\{x : u(x) \geq c\}$ and $\{x : u(x) \leq c\}$ as preference contour sets.

2. WARP and the importance of the observed menus**25 points**

Let $X = \{a, b, c\}$. Initially, the observed menus are all singletons and all pairs, but not X itself. The choice rule selects the sole element of each singleton and satisfies

$$C(\{a, b\}) = \{a\}, \quad C(\{b, c\}) = \{b\}, \quad C(\{a, c\}) = \{c\}.$$

Use the handout's definition of WARP: for $x, y \in B \cap B'$,

$$x \in C(B), y \in C(B') \implies x \in C(B').$$

A preference *rationalizes* a choice rule when its set of best elements is *exactly* $C(B)$ on each observed menu.

- (a) [5] Does this rule satisfy WARP on the observed domain? Can a complete, transitive preference rationalize it? Prove both answers.
- (b) [7] Now add the menu $\{a, b, c\}$ without changing any pairwise choice. Show that no nonempty choice from the new menu can preserve WARP.
- (c) [6] Instead, change only the original choice from $\{a, c\}$ to $\{a\}$. Find the unique choice from $\{a, b, c\}$ compatible with WARP, and give a utility function rationalizing the resulting rule on every nonempty subset of X .
- (d) [7] More generally, prove that the rule $C(B) = \arg \max_{x \in B} u(x)$ satisfies WARP for any finite set of alternatives and any collection of its nonempty menus. Does the same conclusion hold for an arbitrary *single-valued selection* from these maximizer sets when there are ties? Either prove it or give a counterexample using two distinct menus.

3. Cobb–Douglas demand with a corner solution**25 points**

A consumer has consumption set $\mathbb{R}_+^2$ and utility

$$u(x_1, x_2) = x_1^\alpha (x_2 + b)^{1-\alpha}, \quad 0 < \alpha < 1, \quad b > 0.$$

Prices satisfy $p_1, p_2 > 0$, income is $I > 0$, and the budget is $p_1 x_1 + p_2 x_2 \leq I$. Use budget substitution and one-variable calculus; no constrained-programming theorem is needed.

- (a) **[5]** Explain why an optimum has $x_1 > 0$ and exhausts the budget. Compute MRS_{12} at bundles with $x_1 > 0$, and explain why taking the logarithm of utility preserves the maximizer on this region.
- (b) **[9]** Find the unique Marshallian demand for *all* $I > 0$. Identify the income threshold separating $x_2 = 0$ from $x_2 > 0$. Justify global optimality, not merely the first-order condition, and check continuity at the threshold.
- (c) **[6]** Calculate both income derivatives in each open regime and explain whether either good is inferior.
- (d) **[5]** Set $\alpha = 1/2$, $b = 2$, and $p_1 = p_2 = 1$. Find demand at $I = 1$ and $I = 4$, and compare the MRS with p_1/p_2 at each optimum. Explain why imposing tangency at every income would give an incorrect answer.

4. A medical subsidy or an equally costly cash transfer?

25 points

Good 1 is medical care and good 2 is other consumption. Market prices are $p_1, p_2 > 0$ and income is $I > 0$. A per-unit subsidy $s \in (0, p_1)$ lowers the consumer's price of medical care to $q = p_1 - s$. Let x^s maximize utility under the subsidized budget and suppose it exhausts that budget. The government's cost is $G = sx_1^s$. A proposed alternative removes the subsidy and gives the consumer the *fixed* cash transfer G , independently of the new purchases. Treat this as a consumer-welfare and government-cost comparison; there are no externalities or administrative costs in the exercise.

- (a) [5] Show that the consumer is weakly better off with the cash transfer. Does this argument require convex preferences? Does it imply a strictly higher utility level in every case?
- (b) [7] Give a utility function for which the consumer is exactly as well off under the two policies, and verify the claim. Then show that the gain is strict if x^s has both coordinates positive and utility is differentiable near x^s with positive marginal utilities. *Hint:* Consider a small feasible move along the cash-transfer budget line.
- (c) [7] Now take $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, $0 < \alpha < 1$. Find x^s , G , and the cash-transfer optimum x^c . Show that medical consumption is lower under cash while utility is higher.
- (d) [6] For these Cobb–Douglas preferences, find the *smallest* cash transfer T^* that, at market prices, maintains the subsidy utility level. Prove $0 < T^* < G$. Evaluate G and T^* when $\alpha = 1/2$, $p_1 = 2$, $p_2 = 1$, $I = 120$, and $s = 1$.