

Lecture Note on Consumption Theory

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1 Rational Choice

We model a consumer's behaviour by specifying a preference relation on the set of alternatives. Let X denote the set of all possible alternatives (the choice set). In consumer theory, X is typically a subset of $\mathbb{R}_+^N$, representing bundles of N goods.

Definition 1.1 (Preference relation). A preference relation $\succsim$ is a binary relation on X . We read $x \succsim y$ as “the agent weakly prefers x to y .” Two derived relations are defined as follows:

- Strict preference: $x \succ y$ if $x \succsim y$ and not $y \succsim x$.
- Indifference: $x \sim y$ if $x \succsim y$ and $y \succsim x$.

In economics, we elicit $\succsim$ through incentivised choices rather than self-reported surveys, since incentivised choices reveal preferences more reliably. An agent chooses x from $S \subset X$ if x is the $\succsim$ -best element of S .

Definition 1.2 (Rational preference relation (MWG Def. 1.B.1)). A preference relation $\succsim$ on X is rational if it satisfies:

- (i) **Completeness**: For all $x, y \in X$, $x \succsim y$ or $y \succsim x$ (or both).
- (ii) **Transitivity**: For all $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$.

Interpretation of rationality

When X is finite, completeness and transitivity together guarantee that the agent can rank all alternatives from best to worst. Completeness rules out “I cannot compare these”; transitivity rules out preference cycles.

Remark 1. *Completeness of $\succsim$ implies asymmetry of $\succ$ (i.e., there exist no x, y such that $x \succ y$ and $y \succ x$).*

Remark 2 (Limits of transitivity). *The transitivity of $\succsim$ implies transitivity of $\sim$. This can be unreasonable: let x_i be a cup of coffee with i grams of sugar. A consumer cannot detect a 1-gram difference, so $x_i \sim x_{i+1}$ for every i . Transitivity would then imply $x_1 \sim x_{100}$ —but 100 extra grams is clearly detectable. Rationality is therefore a modelling idealisation rather than a literal description of cognition.*

1.1 Choice Structures and Choice Rules (MWG §1.B–1.C)

So far we have described a consumer's preference directly via the preference relation $\succsim$. An alternative—and complementary—approach is to model behaviour through a choice rule, asking: given a set of feasible alternatives, what does the agent actually choose?

Definition 1.3 (Choice structure). A choice structure is a pair $(\mathcal{B}, C(\cdot))$ where:

- (i) $\mathcal{B}$ is a collection of nonempty subsets of X (the possible choice sets or budget sets);

- (ii) $C(\cdot)$ is a choice rule: a mapping that assigns to each $B \in \mathcal{B}$ a nonempty set $C(B) \subseteq B$ (the chosen elements from B).

Interpretation

$C(B)$ is the set of alternatives the agent might select when the feasible set is B . Allowing $C(B)$ to be a set (rather than a single element) captures indifference: if x and y are both in $C(B)$, the agent is willing to choose either.

Example 1 (Walrasian budget sets). In consumer theory, $X = \mathbb{R}_+^L$. For prices $p \in \mathbb{R}_{++}^L$ and wealth $w > 0$, define the Walrasian budget set

$$B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}.$$

The family $\mathcal{B} = \{B_{p,w} : p \in \mathbb{R}_{++}^L, w > 0\}$ is the standard collection of budget sets in the consumer's problem. The Walrasian demand correspondence $x(\cdot, \cdot)$ defines a choice rule via $C(B_{p,w}) = x(p, w)$.

The Weak Axiom of Revealed Preference for Choice Structures

Definition 1.4 (Weak Axiom of Revealed Preference — abstract form (MWG Def. 1.C.1)). The choice structure $(\mathcal{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (WARP) if the following holds: for any $B, B' \in \mathcal{B}$ and any $x, y \in B \cap B'$,

$$x \in C(B) \text{ and } y \in C(B') \implies x \in C(B').$$

Note that the requirements depend on the sets $\mathcal{B}$: As $\mathcal{B}$ becomes richer, the requirements become stronger.

Intuition for WARP

If the agent chooses x from B (when y was also available), then x is revealed preferred to y . WARP says: whenever y is later chosen from some B' that also contains x , the agent must be willing to choose x from B' as well. In other words, the agent cannot simultaneously reveal $x \succ y$ and $y \succ x$.

The Relationship Between Preferences and Choice Rules (MWG §1.C)

Definition 1.5 (Rationalization). A choice structure $(\mathcal{B}, C(\cdot))$ is rationalized by a preference relation $\succsim$ if

$$C(B) = C_{\succsim}(B) \equiv \{x \in B : x \succsim y \text{ for all } y \in B\} \quad \text{for every } B \in \mathcal{B}.$$

That is, the chosen set equals the set of $\succsim$ -maximal elements of B .

Proposition 1 (Rationality implies WARP (MWG Prop. 1.C.1)). If $\succsim$ is a rational preference relation and $(\mathcal{B}, C_{\succsim}(\cdot))$ is the induced choice structure, then $(\mathcal{B}, C_{\succsim}(\cdot))$ satisfies WARP.

Proof. Let $B, B' \in \mathcal{B}$ and $x, y \in B \cap B'$. Suppose $x \in C_{\succsim}(B)$ and $y \in C_{\succsim}(B')$.

- $x \in C_{\succsim}(B)$ implies $x \succsim z$ for all $z \in B$; in particular, $x \succsim y$ (since $y \in B$).
- $y \in C_{\succsim}(B')$ implies $y \succsim z$ for all $z \in B'$; in particular, $y \succsim x$ (since $x \in B'$).
- By completeness and transitivity, $x \succsim y$ and $y \succsim x$ give $x \sim y$. Since $y \succsim z$ for all $z \in B'$ and $x \sim y$, we have $x \succsim z$ for all $z \in B'$ (by transitivity: $x \sim y \succsim z$ implies $x \succsim z$).
- Therefore $x \in C_{\succsim}(B')$. □

Proposition 2 (WARP implies rationalizability (MWG Prop. 1.C.2)). *Suppose $\mathcal{B}$ contains all nonempty subsets of X of size at most three. If $(\mathcal{B}, C(\cdot))$ satisfies WARP, then there exists a unique rational preference relation $\succsim$ that rationalizes $(\mathcal{B}, C(\cdot))$.*

Proof. Define a relation $\succeq$ on X by

$$x \succeq y \iff x \in C(\{x, y\}).$$

Since choices are nonempty, this relation is reflexive and complete.

Step 1: The relation rationalizes the choice rule. We show that, for every $B \in \mathcal{B}$,

$$C(B) = \{x \in B : x \succeq y \text{ for every } y \in B\}.$$

First, let $x \in C(B)$ and take any $y \in B$. Choose $z \in C(\{x, y\})$. Since x and z belong to both B and $\{x, y\}$, WARP implies $x \in C(\{x, y\})$. Thus $x \succeq y$.

Conversely, suppose $x \in B$ and $x \succeq y$ for every $y \in B$. Choose $z \in C(B)$, which exists because choices are nonempty. By the definition of $\succeq$, we have $x \in C(\{x, z\})$. Applying WARP to $\{x, z\}$ and B gives $x \in C(B)$.

Step 2: The relation is transitive. Suppose $x \succeq y$ and $y \succeq z$. The set $T = \{x, y, z\}$ belongs to $\mathcal{B}$, so $C(T)$ is nonempty. We claim that $x \in C(T)$.

If $y \in C(T)$, then $x \in C(\{x, y\})$ and WARP imply $x \in C(T)$. If $z \in C(T)$, then $y \in C(\{y, z\})$ and WARP first imply $y \in C(T)$, and the preceding argument then implies $x \in C(T)$. Since at least one of x , y , and z belongs to $C(T)$, the claim follows.

By Step 1, $x \in C(T)$ implies $x \succeq z$. Hence $\succeq$ is transitive and therefore rational.

Step 3: Uniqueness. Let $\succeq'$ be any rational preference relation that rationalizes the choice rule. Because it rationalizes choices from every binary set,

$$x \succeq' y \iff x \in C(\{x, y\}) \iff x \succeq y.$$

Thus $\succeq' = \succeq$. □

Notice that three-element sets are used only in Step 2 to establish transitivity. Binary sets alone imply uniqueness, conditional on the existence of a rational rationalization. Without the three element sets, WARP does not allow us to conclude the transitivity.

Summary: Preference relations vs. choice rules

- Rational preferences always generate a choice rule satisfying WARP (Proposition above).
- When $\mathcal{B}$ is sufficiently rich (contains all pairs and triples), WARP is sufficient to recover a unique rational preference relation.
- For Walrasian budget sets, however, $\mathcal{B}$ is not rich enough: WARP is necessary but generally not sufficient for rationalizability when $L \geq 3$ goods are present. (See Section 12 for the demand-theoretic version of WARP.)

2 Utility Function

A preference relation $\succsim$ is powerful but abstract. To work with it analytically, we seek a real-valued function that encodes the preference ordering.

Definition 2.1 (Utility function). A function $u : X \rightarrow \mathbb{R}$ represents $\succsim$ if

$$x \succsim y \iff u(x) \geq u(y) \quad \text{for all } x, y \in X.$$

Remark 3 (Ordinal utility). *If u represents $\succsim$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing, then $\phi \circ u$ also represents $\succsim$. Only the ordering of utility values matters, not their magnitude. This is the foundation of ordinal utility theory: utility numbers are a convenient representation, not a cardinal measure of wellbeing.*

Proposition When X is finite, $\succsim$ can be represented by a utility function if and only if $\succsim$ is rational.

Proof. For each $x \in X$, define $W(x)$ as the set of elements in X that x is weakly preferred to. That is,

$$W(x) = \{y \in X \mid x \succsim y\}.$$

Then define $u(x) = |W(x)| = \sum_{y \in W(x)} 1$. □

When X is infinite, even if it is countable, the previous proof does not work because $u(x)$ can be infinity and it can be that $u(x) = +\infty = u(y)$ for $x \succ y$.

Proposition When X is countable, the results also holds: $\succsim$ can be represented by a utility function if and only if $\succsim$ is rational.

Proof. Consider how to construct a utility function. Enumerate X by $\{x_1, x_2, x_3, \dots\}$. Define $u(x) = \sum_{x_i \in W(x)} (\frac{1}{2})^i$. (Now the value is bounded.) □

2.1 Lexicographic preference

When X is uncountable, the result may not be the case. Even if $\succsim$ is rational, it may not have a utility representation.

Example 2 (Lexicographic preference on $\mathbb{R}^2$). For any pairs in $\mathbb{R}^2$, define

$$(x_1, x_2) \succ (y_1, y_2) \quad \text{if } x_1 > y_1, \text{ or } (x_1 = y_1 \text{ and } x_2 > y_2).$$

Let $x \succsim y$ if not $(y_1, y_2) \succ (x_1, x_2)$.

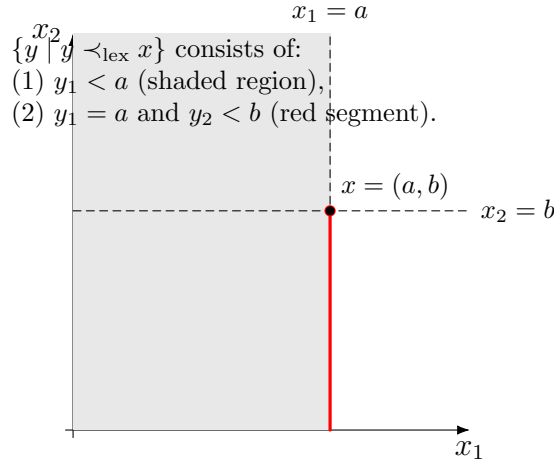
Proposition Lexicographic preference relations are rational. However, for any Lexicographic preference $\succ$ relation, there is no $u : \mathfrak{R}^2 \rightarrow \mathfrak{R}$ such that $x \succ y \implies u(x) > u(y)$.

Proof. It is easy to show that $\succsim$ is rational.

Proof by contradiction. For any $x \in \mathfrak{R}$ such that $(x, 1) \succ (x, 0)$. Then $u(x, 0) < u(x, 1)$. Choose a rational number $r \in (u(x, 0), u(x, 1))$. Denote such r by r_x . For any $x, y \in \mathfrak{R}$ such that $x < y$, I have $u(x, 0) < r_x < u(x, 1) < u(y, 0) < r_y < u(y, 1)$. So $r_x < r_y$. So for each $\mathfrak{R}$, I obtain one to one mapping to $\mathbf{Q}$. This is a contradiction. $\square$

When X is uncountable, for the existence of utility function, we need a technical condition that there exists a countable subset of Y of X such that

$$x \succ y \implies \exists z \in Y \text{ s.t. } x \succsim z \succsim y.$$



2.2 Continuous utility function

For uncountable X (e.g. $X = \mathbb{R}_+^N$), an additional regularity condition is needed.

Definition 2.2 (Continuous preference relation). $\succsim$ is continuous if for every $y \in X$, the upper contour set $\{x \in X : x \succsim y\}$ and the lower contour set $\{x \in X : y \succsim x\}$ are both closed. (This requires a topology on X ; the standard choice is the Euclidean topology.)

Definition 2.3 (Separable topological space). A topological space X is separable if there exists a countable subset $Y \subseteq X$ whose closure equals X .

Definition 2.4 (Connected topological space). A topological space X is connected if there exists no disjoint, nonempty, closed sets A and B such that $A \cup B = X$.¹

The following is a general sufficient condition for the existence of a continuous utility representation.

Theorem 1 (Eilenberg-Debreu). *Let X be a connected and separable topological space. A preference relation $\succeq$ on X is rational and continuous if and only if there exists a continuous utility function $u : X \rightarrow \mathbb{R}$ that represents $\succeq$.*²

For consumer theory, we usually work with subsets of Euclidean space. In this case we can use the following simpler version.

Theorem 2 (Continuous utility representation in Euclidean space). *Let $X \subseteq \mathbb{R}^N$. If $\succeq$ is rational and continuous, then there exists a continuous utility function $u : X \rightarrow \mathbb{R}$ that represents $\succeq$.*

2.3 Monotonic and continuous utility function

- Let $X = \mathbb{R}_+^N$. (Then X is connected.)
- A preference relation $\succsim$ is monotonic if for any $x, y \in X$, $x_i \geq y_i$ for all i implies $x \succsim y$; $x_i \geq y_i$ for all i and $x_j > y_j$ for some j implies $x \succ y$.
- A utility function u is monotonic if $x_i \geq y_i$ for all i implies $u(x) \geq u(y)$; $x_i \geq y_i$ for all i and $x_j > y_j$ for some j implies $u(x) > u(y)$.

Proposition Let $X = \mathbb{R}_+^N$. A preference relation $\succsim$ is complete, transitive, continuous, and monotonic if and only if there exists a monotonic continuous utility function u that represents $\succsim$.

Proof. Fix $x \in X$. Find a number α such that $x \sim (\alpha, \dots, \alpha)$. Define $u(x) = \alpha$.

The purpose of the following proof is to show the existence of such α and its uniqueness. Define $B(x) = \{t \in \mathbb{R}_+ | t(1, \dots, 1) \succsim x\}$ and $W(x) = \{t \in \mathbb{R}_+ | x \succsim t(1, \dots, 1)\}$.

- Show that both $B(x)$ and $W(x)$ are nonempty and closed.
- Show also that $B(x) \cup W(x) = \mathbb{R}_+$.
- Use the fact that $\mathbb{R}_+$ is connected to show $B(x) \cap W(x) \neq \emptyset$. Let α be an element of $B(x) \cap W(x)$.
- By monotonicity, we can show that α is unique and u is monotonic
- By using transitivity, we can show that u represents $\succsim$.
- By using the continuity of $\succsim$, we can show that u is continuous.

□

¹The definition can be stated under the assumption that both A and B are open sets, instead of closed sets.

²Debreu's version assumes a second countable topological space, instead of connected and separable topological space

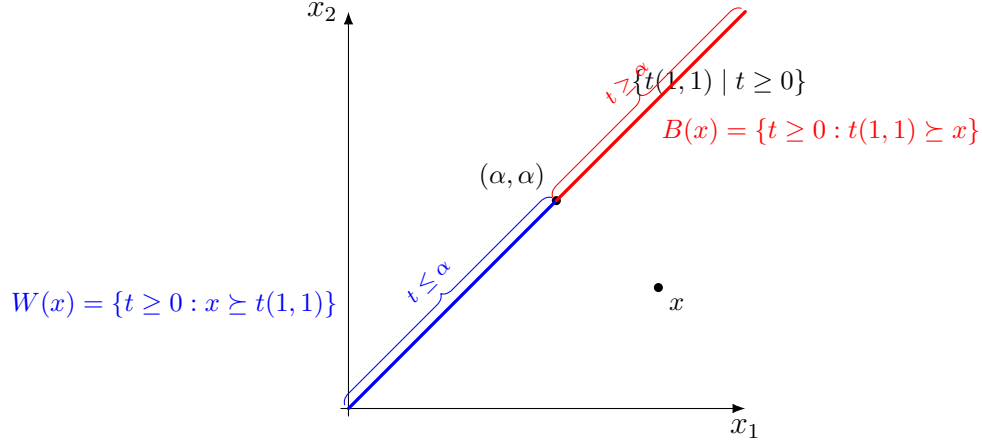


Figure 1: Add a indifference curve that goes through x and (α, α)

2.4 Differentiable utility function

add content later.

3 Indifference Curves

Consider an economy with N goods. The consumption set $X \subseteq \mathbb{R}^N$ is the set of feasible commodity bundles. Throughout, we fix a utility function $u : X \rightarrow \mathbb{R}$ representing the consumer's preferences.

Definition 3.1 (Indifference curve and upper contour set). For a utility level $a \in \mathbb{R}$, the indifference curve at level a is

$$\mathcal{I}(a) = \{x \in X : u(x) = a\},$$

and the upper contour set is $U(a) = \{x \in X : u(x) \geq a\}$.

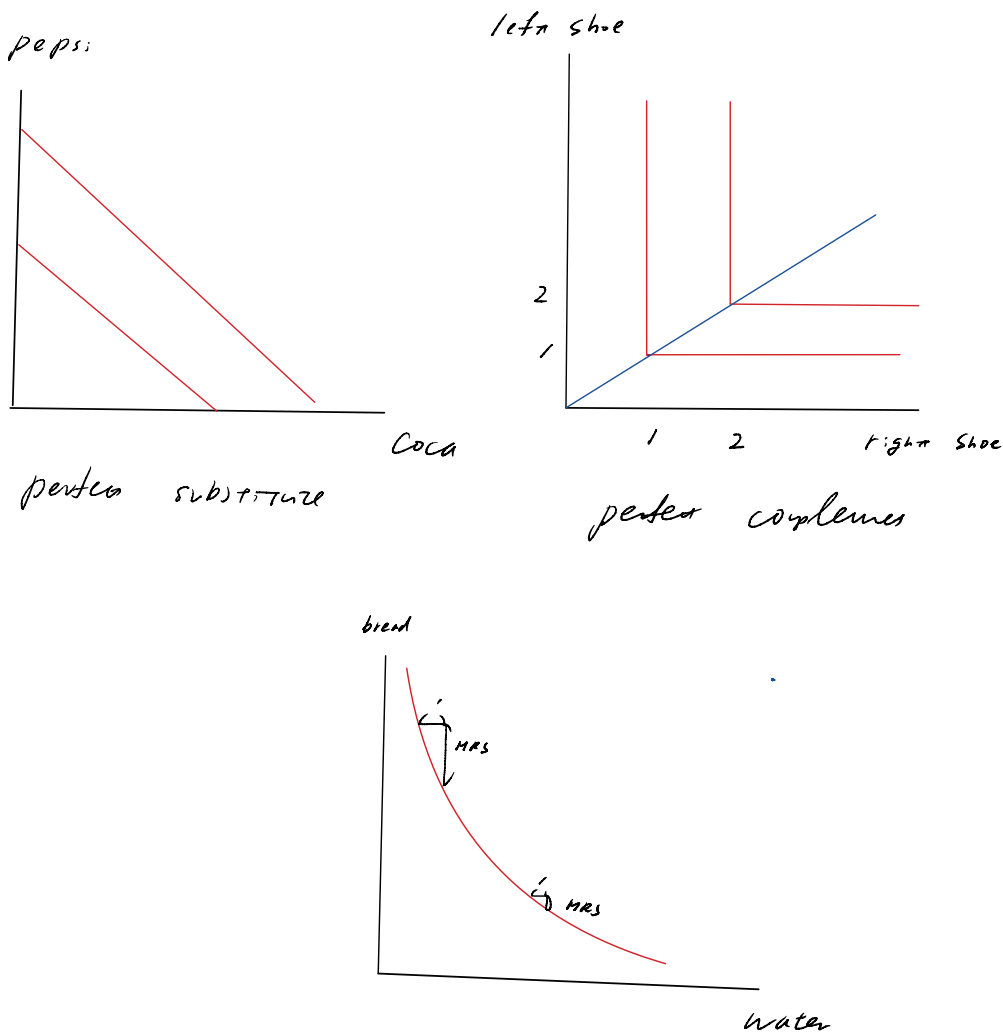
Two polar cases illustrate the range of possible preference structures.

Example 3 (Perfect substitutes). *Coke and Pepsi for a consumer who is indifferent between them: $u(x_1, x_2) = x_1 + x_2$. Indifference curves are straight lines with slope -1 .*

Example 4 (Perfect complements). *Left and right shoes: $u(x_1, x_2) = \min\{x_1, x_2\}$. Indifference curves are L-shaped.*

In most economic applications, indifference curves lie between these two extremes and exhibit a diminishing marginal rate of substitution.

Definition 3.2 (Marginal Rate of Substitution: intuitive definition). The marginal rate of substitution of good 1 for good 2 at a bundle x , denoted $MRS_{12}(x)$, is the rate at which the consumer is willing to give up good 2 per unit of additional good 1 while remaining on the same indifference curve. (We provide a formal definition later.)



4 Marginal Utility and MRS

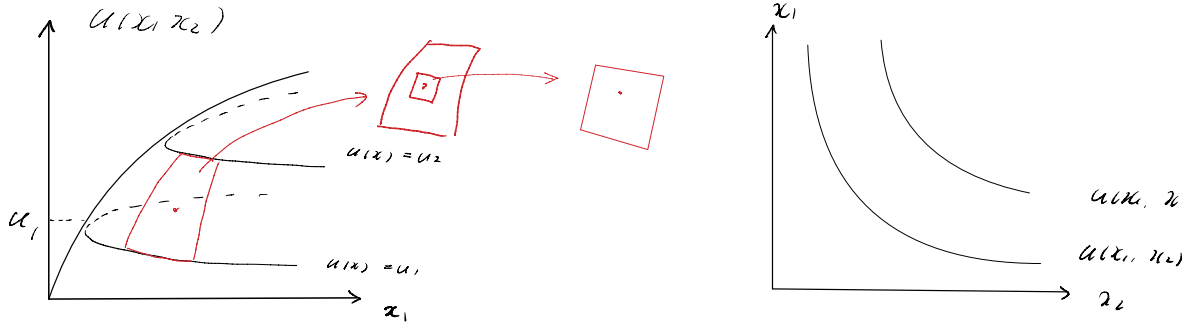
Let $X \subseteq \mathbb{R}^N$ be the consumption set and let $u : X \rightarrow \mathbb{R}$ be a differentiable utility function. Write $x = (x_1, \dots, x_N)$.

Definition 4.1 (Marginal utility). For each good $i \in \{1, \dots, N\}$, the marginal utility of good i at bundle x is

$$MU_i(x) := \frac{\partial u(x)}{\partial x_i}.$$

MU_i : marginal increase of utility when the consumption of good i increases marginally.

Remark 4 (First-order approximation / local linearization). If u is differentiable at x and



the bundle changes by a small vector $\Delta x \in \mathbb{R}^N$, then

$$u(x + \Delta x) - u(x) \approx \sum_{i=1}^N \frac{\partial u(x)}{\partial x_i} \Delta x_i = \nabla u(x) \cdot \Delta x.$$

This is the multivariate analogue of the tangent-line approximation in one variable.

Intuitive Explanation. If the consumption of good i alone increases by Δx_i , utility changes by approximately $\Delta u \approx \frac{\partial u}{\partial x_i} \Delta x_i$. When several goods change simultaneously (good i increases by Δx_i), the linear case $u(x) = \sum_i a_i x_i$ gives $\Delta u = \sum_i a_i \Delta x_i$ exactly. For a smooth u , the function looks approximately linear locally, so the same formula holds to first order:

Total differential approximation

$$\Delta u \approx \sum_{i=1}^N \frac{\partial u}{\partial x_i} \Delta x_i.$$

Taking derivatives thus lets us analyse a local linear model of any smooth utility function.

Proposition 3 (Total differential). If u is differentiable, then along any differentiable path $t \mapsto x(t)$,

$$du = \sum_{i=1}^N \frac{\partial u}{\partial x_i} dx_i.$$

4.1 MRS and marginal utility

Let $N = 2$. Along a change (dx_1, dx_2) , the total differential is

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2.$$

Assuming $du = 0$ (i.e., the same indifference curve), which is nonzero by assumption,

$$MRS_{12}(v) = -\frac{dx_2}{dx_1} = \frac{\partial u / \partial x_1}{\partial u / \partial x_2}.$$

Definition 4.2 (Marginal rate of substitution). Fix a utility level $\bar{u}$ and consider the indifference curve $\{x \in X : u(x) = \bar{u}\}$. At a differentiable point x on this curve with $MU_2(x) \neq 0$, the marginal rate of substitution of good 1 for good 2 is

$$MRS_{12}(x) := -\frac{dx_2}{dx_1} \Big|_{du=0} = \frac{MU_1(x)}{MU_2(x)}.$$

Proposition 4 (MRS is ordinal). *Let $v = \phi \circ u$ where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and strictly increasing. Then at any x with $MU_2(x) \neq 0$,*

$$MRS_{12}^v(x) = MRS_{12}^u(x).$$

Proof. By the chain rule, $\frac{\partial v}{\partial x_i} = \phi'(u(x)) \frac{\partial u}{\partial x_i}$ for $i = 1, 2$. Therefore

$$\frac{\partial v / \partial x_1}{\partial v / \partial x_2} = \frac{\phi'(u(x)) \partial u / \partial x_1}{\phi'(u(x)) \partial u / \partial x_2} = \frac{\partial u / \partial x_1}{\partial u / \partial x_2}.$$

□

Definition 4.3 (Law of diminishing MRS). The law of diminishing marginal rate of substitution states that MRS_{12} decreases as x_1 increases along an indifference curve. Geometrically, this is equivalent to the indifference curves being convex (bowing toward the origin).

Remark 5 (Why convexity is a reasonable assumption). *If indifference curves are not convex, demand may respond discontinuously to price changes. Empirically, observed demand tends to move continuously with prices, which is consistent with convexity. Moreover, even when true indifference curves are non-convex, a convexification (taking the convex hull of upper contour sets) leaves the Walrasian demand function nearly unchanged, so convexity is approximately without loss of generality.*

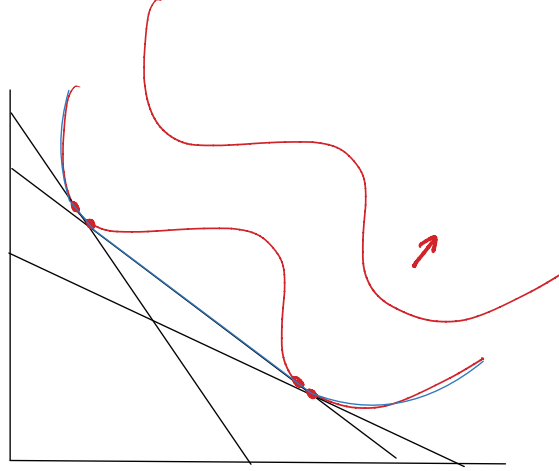
4.2 Convex preferences and the law of decreasing MRS

We say that $\succsim$ is *convex* if $\{y \in X | y \succsim x\}$ is convex for any $x \in X$.³

Remark 6 (Convex preferences $\iff$ diminishing MRS (smooth 2D intuition)). *Assume there is a C^2 utility function $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ representing $\succ$, with $u_1 > 0$ and $u_2 > 0$. Write the marginal rate of substitution as*

$$MRS(x) := \frac{u_1(x)}{u_2(x)}.$$

³Alternatively, we can define convexity as $\{y \in X | y \succ x\}$ is convex for any x , assuming that $\succsim$ is complete and transitive and $x \succ y$ if and only if not $y \succsim x$.



Then (locally) $\succ$ is convex (equivalently, every upper contour set $\{u \geq k\}$ is convex) if and only if the MRS is diminishing along indifference curves:

$$\frac{d}{dx_1} \text{MRS}(x_1, g(x_1)) \leq 0,$$

where $x_2 = g(x_1)$ is a local parametrization of an indifference curve.

Intuition. With $u_2 > 0$, each indifference curve can be written as a smooth graph $x_2 = g(x_1)$, and the upper contour set is simply the region above it: $\{x_2 \geq g(x_1)\}$. Convex preferences mean these “regions above” are convex, which in turn means the boundary curve is convex (i.e. it bends inward), so $g'' \geq 0$. But the slope of the indifference curve is the negative MRS:

$$g'(x_1) = -\text{MRS}(x_1, g(x_1)).$$

So if the curve bends inward ($g'' \geq 0$), its slope becomes less negative as x_1 increases, meaning $-\text{MRS}$ increases, i.e. MRS decreases. Conversely, if MRS decreases along the indifference curve, then $g' = -\text{MRS}$ increases, so the curve bends inward and the upper contour set is convex.

4.3 Convexity of $\succ$ if and only if quasi-concave utility

Assume for simplicity that X is ordered and convex set.

We say a function f is *quasi-concave* if for any $x, y \in X$ and $\alpha \in [0, 1]$ such that

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}.$$

We say a function f is *concave* if for any $x, y \in X$ and $\alpha \in [0, 1]$ such that

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y).$$

Proposition Assume that $\succsim$ is complete and transitive and a function u represents $\succsim$. $\succsim$ is convex if and only if u is quasi concave, that is

Proof.

Step 1: If $\succsim$ is convex then u is quasi concave.

Take $x, y \in X$ and a number α to show that u is quasi concave. WLOG, assume that $u(x) \geq u(y)$. Then $\alpha x + (1 - \alpha)y \in \{x \in X \mid x \succsim y\}$. Thus we have

$$u(\alpha x + (1 - \alpha)y) \geq u(y) = \min\{u(x), u(y)\}.$$

Step 2: If u is quasi-convex, then $\succsim$ is convex.

Fix $z \in X$. Take $x, y \in X$ such that $x \succsim z$ and $y \succsim z$. Take a number α to show that $\alpha x + (1 - \alpha)y \succsim z$, which holds because

$$u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\} \geq u(z).$$

□

Remark The convexity of a preference relation does not necessarily implies an existence of a concave utility function, even though a quasi-concave utility always exists. In particular, we give an example of convex set X and a complete, transitive, and convex relation $\succsim$ such that for every representation u , the function u is not concave, that is, there exist some points $x, y \in X$ and $\alpha \in (0, 1)$ such that

$$u(\alpha x + (1 - \alpha)y) < \alpha u(x) + (1 - \alpha)u(y).$$

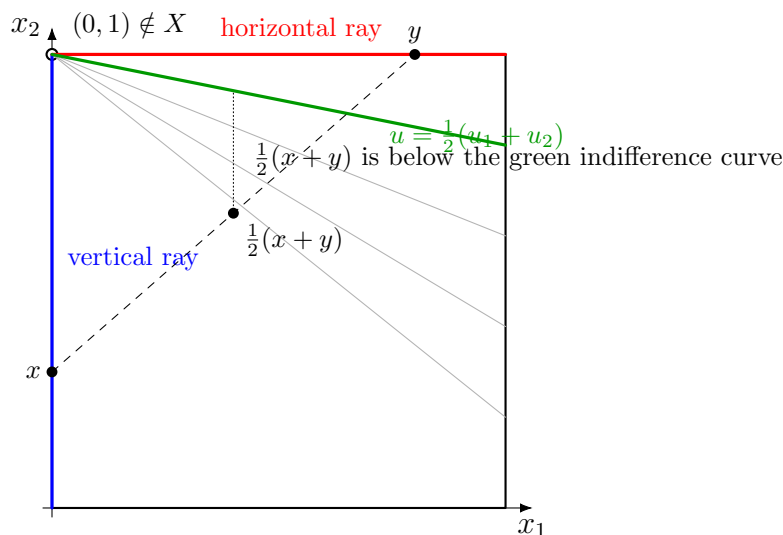
Proof. Let $X = ([0, 1] \times [0, 1]) \setminus \{(0, 1)\}$. The set is convex. The indifference curves are rays emerging from the point $(0, 1)$, and the utility increases as a point moves to the northeast direction. Thus $\succsim$ is convex.

Fix any utility representation u . Denote by u_1 the representation for points on the vertical ray (blue), and by u_2 the representation for points on the horizontal ray (red). Choose $\alpha = \frac{1}{2}$. Find the indifference curve (green) whose utility level is $\frac{1}{2}(u_1 + u_2)$. Notice that this indifference curve may vary depending on the chosen representation u .

It is easy to find a point x on the vertical ray and a point y on the horizontal ray such that the convex combination $\frac{1}{2}(x + y)$ lies below the green indifference curve. That is,

$$u\left(\frac{1}{2}(x + y)\right) < \frac{1}{2}(u(x) + u(y)).$$

Thus, u is not concave. Importantly, such points x and y can be found for every possible representation u of $\succeq$. Therefore, $\succeq$ does not admit a concave representation. □



5 Optimal Consumption: Geometric Approach

The consumer's problem is to choose the bundle $x \in X$ that maximizes utility subject to the budget constraint. With N goods at prices $p = (p_1, \dots, p_N)$ and income $I > 0$, the problem is

$$\max_{x \in X} u(x) \quad \text{subject to} \quad p \cdot x \leq I.$$

In the two-good case, the budget line $p_1x_1 + p_2x_2 = I$ can be written as

$$x_2 = -\frac{p_1}{p_2} x_1 + \frac{I}{p_2},$$

a line with slope $-p_1/p_2$ and vertical intercept I/p_2 . The consumer maximizes utility by reaching the highest attainable indifference curve subject to the budget constraint. At an interior optimum, the indifference curve is tangent to the budget line, giving the condition:

Tangency condition (geometric optimality)

At an interior optimum x^* ,

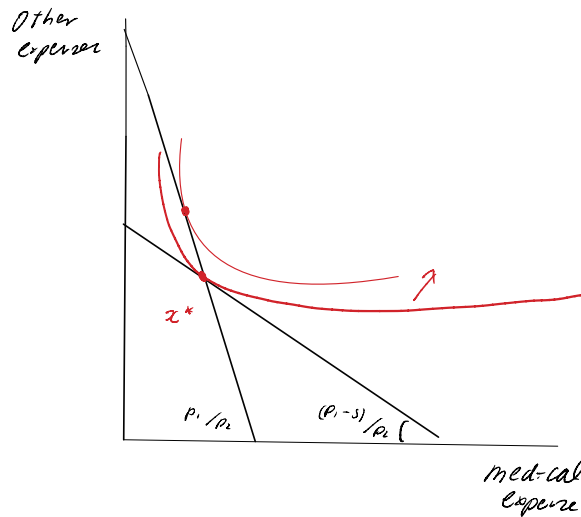
$$MRS_{12}(x^*) = \frac{p_1}{p_2}.$$

The marginal rate of substitution equals the price ratio. Intuitively: if $MRS_{12} > p_1/p_2$, the consumer values good 1 more than the market does, so it is optimal to consume more of good 1 (and less of good 2) until equality is restored.

5.1 Example: Subsidy to medical expenses of elderly people.

In Japan, people older than 70 pay only 10% of their medical expenses; the other 90% is paid by the government (tax payer).

Is there alternative policy that costs less and does not make the elderly person unhappy.



Let x_1 denote medical expenditure and x_2 all other expenditure. Under a per-unit subsidy of S per unit of medical expense, the elderly consumer faces the subsidised budget constraint

$$(p_1 - S)x_1 + p_2 x_2 = I.$$

Let $x^* = (x_1^*, x_2^*)$ be the optimal bundle under the subsidy: $x^* \in \arg \max u(x_1, x_2)$ subject to $(p_1 - S)x_1 + p_2 x_2 = I$. The government spends Sx_1^* on the subsidy.

Now consider an alternative policy: abolish the subsidy and instead transfer the cash amount Sx_1^* directly to the consumer. The consumer's new budget set is $\{(x_1, x_2) : p_1 x_1 + p_2 x_2 \leq I + Sx_1^*\}$, which contains the subsidised optimum x^* (since $p_1 x_1^* + p_2 x_2^* = I + Sx_1^*$) and also offers bundles previously unaffordable. The consumer is therefore weakly better off under the cash transfer, at the same government cost.

Efficiency of cash transfers over price subsidies

Distorting market prices (via a subsidy or tax) leads to a smaller total welfare than an equivalent lump-sum cash transfer. A free-market allocation maximizes total welfare; price distortions introduce a deadweight loss.

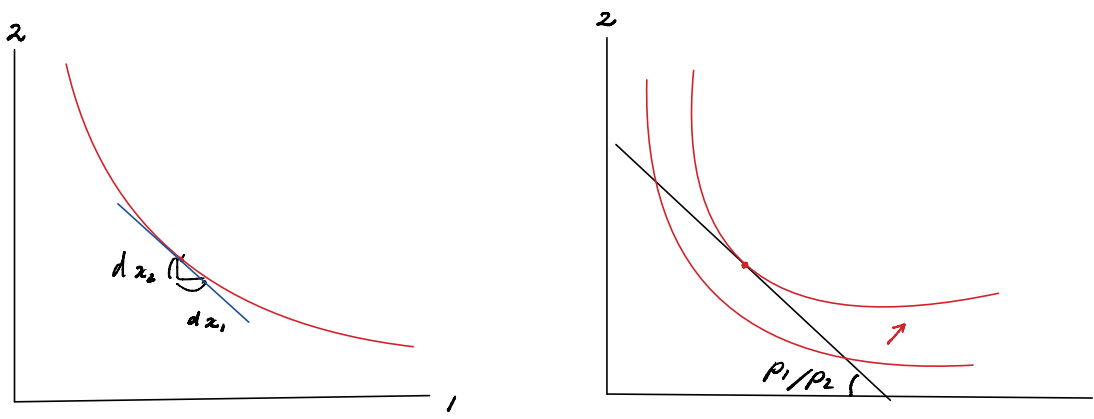
Remark 7 (Equity considerations). *The efficiency argument for cash transfers does not address inequality. How to distribute the total welfare gain involves value judgements that economics, as a positive science, does not resolve. Also, to make the cash transfer, we need to make the elderly person correctly report one's medical spending. However, the elderly person has an incentive to over-report the spending. This is a topic studied in game theory.*

6 Optimal consumption via marginal analysis

Two goods Case: By using the expression for MRS_{12} , we can write the condition for optimality as follows

Key equation

$$\frac{\partial u / \partial x_1}{\partial u / \partial x_2} = MRS_{12} = \frac{p_1}{p_2}.$$



General Case: Let prices be $p = (p_1, \dots, p_N) \in \mathbb{R}_{++}^N$ and income be $I > 0$. The budget set is $\{x \in X : p \cdot x \leq I\}$.

Proposition 5 (Lagrangian first-order conditions). *Consider an interior optimum $x^* \in \text{int}(X)$ of*

$$\max_{x \in X} u(x) \quad \text{s.t.} \quad p \cdot x \leq I,$$

where u is differentiable. Then there exists a scalar $\lambda \geq 0$ such that for each good i ,

$$MU_i(x^*) = \lambda p_i.$$

Equivalently,

$$\frac{MU_i(x^*)}{p_i} = \lambda \quad \text{for all } i.$$

Remark 8 (Law of equal marginal utility (per dollar)). The term $\frac{MU_i(x)}{p_i}$ is the marginal utility gained by spending one more unit of currency on good i . At an interior optimum, the last dollar spent must yield the same marginal utility regardless of which good it is spent on. Notice that

$$\frac{\partial u / \partial x_i}{p_i} = \underbrace{MU_i(x)}_{\text{(marginal utility of good } i)} \times \underbrace{\frac{1}{p_i}}_{\text{(the amount of goods } i \text{ can be purchased by one unit of currency)}}$$

is the additional marginal utility by increasing the spending for consumption of good i marginally.

6.1 Numerical example: Cobb-Douglas utility function

Consider $u(x, y) = x^a y^b$ with $a, b > 0$. The Lagrangian for the UMP is

$$\mathcal{L}(x, y, \lambda) = x^a y^b + \lambda(I - p_x x - p_y y).$$

The first-order conditions are

$$0 = \frac{\partial \mathcal{L}}{\partial x} = ax^{a-1}y^b - \lambda p_x \quad \text{and} \quad 0 = \frac{\partial \mathcal{L}}{\partial y} = x^a by^{b-1} - \lambda p_y.$$

Dividing these gives $ay/(bx) = p_x/p_y$, or equivalently

$$\frac{a}{b} = \frac{p_x x}{p_y y} \equiv \frac{\text{spending on } x}{\text{spending on } y}.$$

Thus the consumer allocates a constant fraction $a/(a+b)$ of income to good x and $b/(a+b)$ to good y . Substituting into the budget constraint $p_x x + p_y y = I$ yields the Marshallian demand:

Cobb-Douglas demand

$$x^*(p, I) = \frac{a}{a+b} \frac{I}{p_x} \quad \text{and} \quad y^*(p, I) = \frac{b}{a+b} \frac{I}{p_y}.$$

Both goods are normal ($\partial x^*/\partial I > 0$), and the expenditure share of each good is constant regardless of prices or income.

7 Properties of Optimal Consumption: The Demand Function

Consider a consumer with a continuous, strictly quasi-concave utility function $u : \mathbb{R}_+^N \rightarrow \mathbb{R}$ facing prices $p = (p_1, \dots, p_N) \in \mathbb{R}_{++}^N$ and income $I > 0$. The consumer's problem is

$$\max_{x \in \mathbb{R}_+^N} u(x) \quad \text{subject to} \quad p \cdot x \leq I. \quad (1)$$

Under the maintained assumptions, the solution to (1) is unique for every (p, I) .

Definition 7.1 (Marshallian demand function). The Marshallian (or ordinary) demand function is the mapping

$$x^* : \mathbb{R}_{++}^N \times \mathbb{R}_{++} \rightarrow \mathbb{R}_+^N, \quad (p, I) \mapsto x^*(p, I),$$

where $x^*(p, I)$ is the unique solution to (1).

7.1 Basic Properties of Walrasian Demand (MWG §3.D)

Two fundamental restrictions on demand follow immediately from the structure of the UMP.

Proposition 6 (Homogeneity of degree zero (MWG Prop. 3.D.2)). *If $x^*(p, I)$ is derived from utility maximization, then for all (p, I) and all $\lambda > 0$,*

$$x^*(\lambda p, \lambda I) = x^*(p, I).$$

Proof. The budget set at $(\lambda p, \lambda I)$ is $\{x \in \mathbb{R}_+^N : \lambda p \cdot x \leq \lambda I\} = \{x \in \mathbb{R}_+^N : p \cdot x \leq I\}$, which is identical to the budget set at (p, I) . Since the feasible set is unchanged, the utility-maximising bundle is unchanged. $\square$

No money illusion

Homogeneity of degree zero says that only relative prices and real income matter. Doubling all prices and income simultaneously leaves the budget set — and hence optimal demand — exactly unchanged. This rules out money illusion: a rational consumer does not change behaviour when all nominal values scale by the same factor. A practical consequence: differentiating $x^*(\lambda p, \lambda I) = x^*(p, I)$ with respect to λ at $\lambda = 1$ (Euler's theorem) gives

$$\sum_{j=1}^N \frac{\partial x_i^*}{\partial p_j} p_j + \frac{\partial x_i^*}{\partial I} I = 0 \quad \text{for each } i.$$

Proposition 7 (Walras' law (MWG Prop. 3.D.3)). *If u is monotonically increasing and $x^*(p, I)$ solves the UMP, then*

$$p \cdot x^*(p, I) = I.$$

Proof. Suppose $p \cdot x^*(p, I) < I$. By local nonsatiation, there exists a bundle y arbitrarily close to x^* with $u(y) > u(x^*)$. For y close enough to x^* , the constraint $p \cdot y \leq I$ is still satisfied. But then y is feasible and achieves strictly higher utility than x^* , contradicting optimality. $\square$

The budget is fully spent

Walras' law says the consumer exhausts the entire income I at the optimum. Differentiating $p \cdot x^*(p, I) = I$ with respect to p_j and I yields two useful identities:

$$\sum_{i=1}^N p_i \frac{\partial x_i^*}{\partial p_j} + x_j^* = 0 \quad (\text{price differentiation}),$$

$$\sum_{i=1}^N p_i \frac{\partial x_i^*}{\partial I} = 1 \quad (\text{income differentiation}).$$

The second identity says that a one-dollar increase in income raises total spending by exactly one dollar.

7.2 Income Effects

Holding prices fixed, an increase in income shifts the budget set outward. The qualitative response of demand to this shift classifies goods as follows.

Definition 7.2 (Normal and inferior goods). Good i is a normal good at (p, I) if

$$\frac{\partial x_i^*(p, I)}{\partial I} > 0,$$

and an inferior good at (p, I) if

$$\frac{\partial x_i^*(p, I)}{\partial I} < 0.$$

Remark 9. In the Cobb–Douglas example, every good is normal: $\partial x_i^*/\partial I = a_i/(\sum_j a_j) \cdot 1/p_i > 0$. Inferiority requires richer preference structures (e.g., quasi-linear utility).

7.3 Own-Price Effects and the Law of Demand

Definition 7.3 (Law of demand). Good i satisfies the law of demand at (p, I) if

$$\frac{\partial x_i^*(p, I)}{\partial p_i} < 0.$$

Definition 7.4 (Giffen good). Good i is a Giffen good at (p, I) if

$$\frac{\partial x_i^*(p, I)}{\partial p_i} > 0.$$

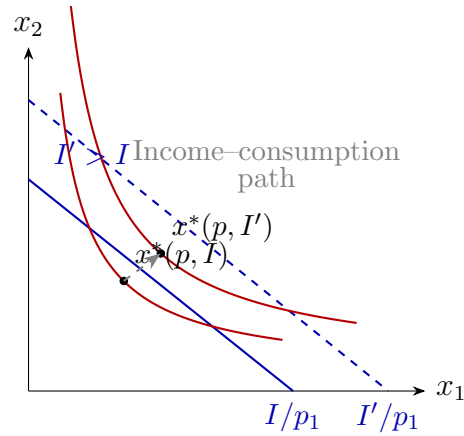
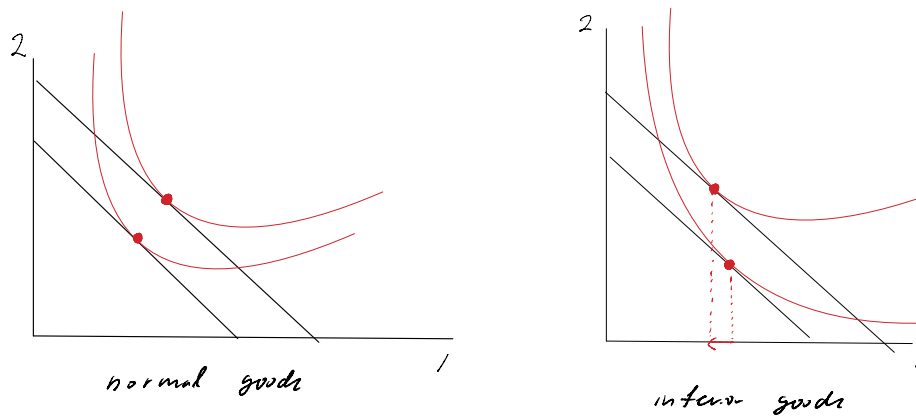
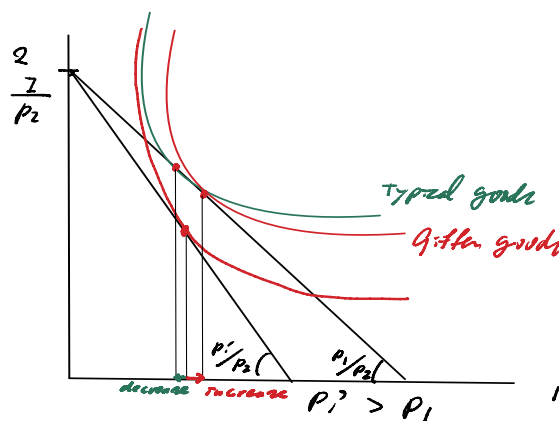


Figure 2: Income effect with two normal goods. As income rises from I to I' , demand for both goods increases.

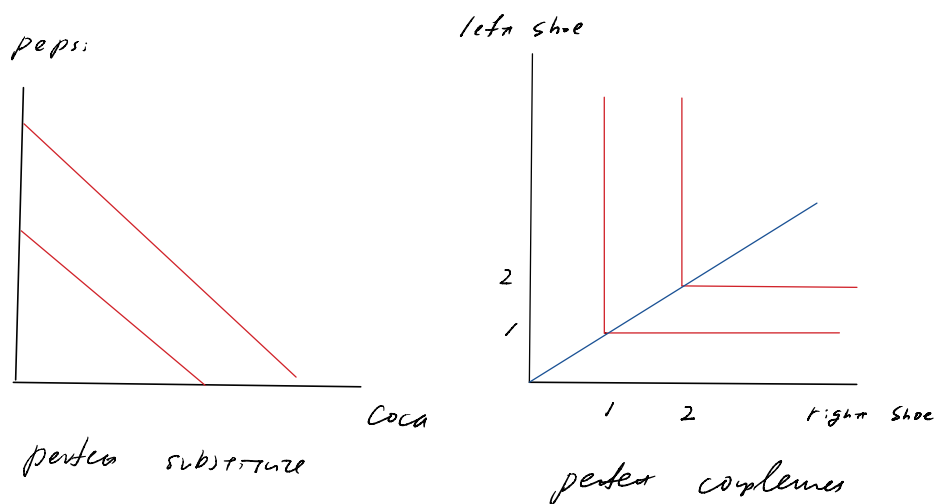


Remark 10. The law of demand does not hold universally for Marshallian demand—Giffen goods are theoretically possible. However, as we show in Section 8, the law of demand does hold for the compensated demand function; the Giffen phenomenon arises only when the income effect is large enough to overwhelm the substitution effect (the Slutsky decomposition makes this precise).



8 Substitution, Complementarity, and the Compensated Demand Function

When the price of one good changes, two conceptually distinct forces act on demand: a substitution effect (the consumer re-optimises along the same indifference curve) and an income effect (the consumer's real purchasing power changes). To isolate the substitution effect, we introduce the compensated (Hicksian) demand function.



Recall the polar cases of perfect substitutes and perfect complements. For most goods, indifference curves lie between these extremes, and a key question is how strongly two goods substitute for or complement each other. The natural tool for measuring this is the compensated (Hicksian) demand function.

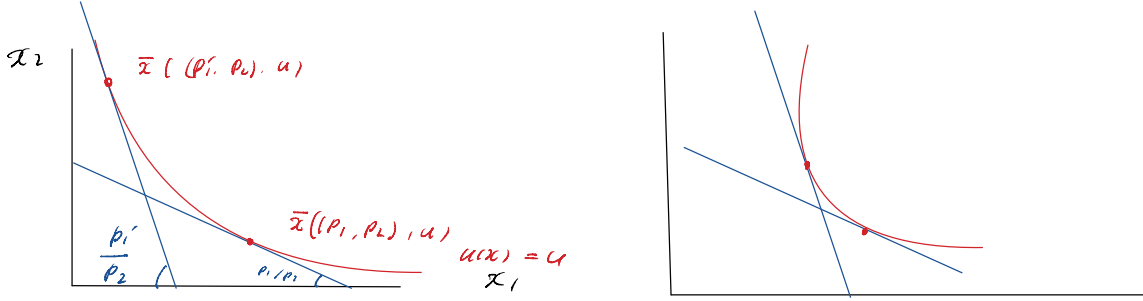
Definition 8.1 (Compensated demand — motivation). Given prices p and a target utility level u , the compensated demand $\bar{x}(p, u)$ is the solution to the expenditure minimisation problem:

Expenditure Minimisation Problem (EMP)

$$\bar{x}(p, u) = \arg \min_x p \cdot x \quad \text{subject to} \quad u(x) \geq u.$$

$\bar{x}(p, u)$ answers: “what is the cheapest bundle that achieves utility u at prices p ?”

To measure substitutability, consider a two-good economy ($N = 2$) and a price increase of good 1 from p_1 to p'_1 . The compensated demands $\bar{x}(p_1, p_2)$ and $\bar{x}(p'_1, p_2)$ both keep utility fixed, so any change in $\bar{x}_2$ reflects pure substitution.



When the two goods are substitutes, $\bar{x}(p_1, p_2)$ and $\bar{x}(p'_1, p_2)$ differ substantially (the consumer switches toward good 2 when good 1 becomes more expensive). When they are complements, the two compensated demands are close (the consumer uses both goods together). A precise measure is the cross-price derivative of compensated demand: if $\partial \bar{x}_2(p_1, p_2)/\partial p_1$ is large and positive, the goods are strong substitutes; if it is negative, they are complements.

Note also that $\partial \bar{x}_1(p_1, p_2)/\partial p_1 < 0$ in the figure: when the own price rises, compensated demand for the good itself falls. This is the negativity of the own-substitution effect, proved formally in Theorem 3.

8.1 The Expenditure Minimization Problem

Definition 8.2 (Compensated demand and expenditure function). Given prices $p \in \mathbb{R}_{++}^N$ and a target utility level $\bar{u}$, define

$$e(p, \bar{u}) \equiv \min_{x \in \mathbb{R}_+^N} p \cdot x \quad \text{subject to} \quad u(x) \geq \bar{u}. \quad (2)$$

The function $e : \mathbb{R}_{++}^N \times \mathbb{R} \rightarrow \mathbb{R}_+$ is the expenditure function, and the solution

$$\bar{x}(p, \bar{u}) \equiv \arg \min_{x \in \mathbb{R}_+^N} \{p \cdot x : u(x) \geq \bar{u}\}$$

is the compensated (Hicksian) demand function.

The expenditure function and compensated demand are linked by

$$e(p, \bar{u}) = p \cdot \bar{x}(p, \bar{u}).$$

Remark 11. *The expenditure function measures the minimum cost of attaining utility $\bar{u}$ at prices p . It is the natural tool for welfare analysis: the change $e(p', \bar{u}) - e(p, \bar{u})$ quantifies the monetary impact of a price change from p to p' , holding welfare constant.*

8.2 General Definitions: Substitutes and Complements

Definition 8.3 (Hicksian substitutes and complements). Goods i and j are (Hicksian) substitutes at $(p, \bar{u})$ if

$$\frac{\partial \bar{x}_j(p, \bar{u})}{\partial p_i} > 0,$$

and (Hicksian) complements at $(p, \bar{u})$ if

$$\frac{\partial \bar{x}_j(p, \bar{u})}{\partial p_i} < 0.$$

Intuitively: if the price of good i rises (with utility held constant), and the consumer responds by increasing compensated demand for good j , then j is a substitute for i (the consumer switches toward j). If compensated demand for j falls, the goods are complements (the consumer reduces j along with i).

Remark 12 (Two-good case). *When $N = 2$, all goods are necessarily substitutes in the Hicksian sense. To see this: if p_1 rises, negativity of self-substitution (Theorem 3) gives $\partial \bar{x}_1 / \partial p_1 \leq 0$, so the consumer demands less of good 1. But utility must be maintained, so compensated demand for good 2 must increase, i.e. $\partial \bar{x}_2 / \partial p_1 > 0$. Complementarity requires $N \geq 3$.*

8.3 Negativity of the Own-Substitution Effect

The following result shows that compensated demand always satisfies the law of demand.

Theorem 3 (Negativity of own-substitution effect). *For every good i ,*

$$\frac{\partial \bar{x}_i(p, \bar{u})}{\partial p_i} \leq 0.$$

Proof. Fix a utility level $\bar{u}$. Let p^0 and p^1 be two price vectors, and let $x^0 = \bar{x}(p^0, \bar{u})$ and $x^1 = \bar{x}(p^1, \bar{u})$ be the corresponding compensated demands.

Step 1. By definition of expenditure minimisation, x^1 minimises cost at prices p^1 subject to $u(x) \geq \bar{u}$. Since x^0 is feasible (it also achieves $\bar{u}$), we have

$$p^1 \cdot x^1 \leq p^1 \cdot x^0.$$

Step 2. By the symmetric argument (exchanging the roles of p^0 and p^1),

$$p^0 \cdot x^0 \leq p^0 \cdot x^1.$$

Step 3. Subtracting the second inequality from the first:

$$(p^1 - p^0) \cdot x^1 = p^1 \cdot x^1 - p^0 \cdot x^1 \leq p^1 \cdot x^0 - p^0 \cdot x^0 = (p^1 - p^0) \cdot x^0,$$

which rearranges to

$$(p^1 - p^0) \cdot (x^1 - x^0) \leq 0. \quad (3)$$

Step 4. Suppose the two price vectors differ only in good i , with $p_i^1 > p_i^0$. Then (3) reduces to

$$(p_i^1 - p_i^0)(\bar{x}_i^1 - \bar{x}_i^0) \leq 0.$$

Since $p_i^1 - p_i^0 > 0$, we conclude $\bar{x}_i^1 - \bar{x}_i^0 \leq 0$, and hence $\partial \bar{x}_i / \partial p_i \leq 0$. $\square$

Remark 13. *The inequality (3) is a general property of compensated demand: the vectors Δp and Δx make an obtuse (or right) angle. This is precisely analogous to the law of demand for factor inputs in production theory.*

9 The Expenditure Function

Definition 9.1 (Expenditure function). The expenditure function $e : \mathbb{R}_{++}^N \times \mathbb{R} \rightarrow \mathbb{R}_+$ is defined by

$$e(p, \bar{u}) = \min_{x \in \mathbb{R}_+^N} \{p \cdot x : u(x) \geq \bar{u}\} = p \cdot \bar{x}(p, \bar{u}).$$

Proposition 8 (Properties of the expenditure function). *Under the maintained assumptions on u , the expenditure function satisfies:*

- (a) **Non-decreasing in p :** $p' \geq p \implies e(p', \bar{u}) \geq e(p, \bar{u})$.
- (b) **Homogeneous of degree 1 in p :** $e(tp, \bar{u}) = t e(p, \bar{u})$ for all $t > 0$.
- (c) **Concave in p :** For all $\lambda \in [0, 1]$,

$$e(\lambda p + (1 - \lambda)p', \bar{u}) \geq \lambda e(p, \bar{u}) + (1 - \lambda) e(p', \bar{u}).$$

- (d) **Shephard's lemma:** $\frac{\partial e(p, \bar{u})}{\partial p_i} = \bar{x}_i(p, \bar{u})$.

- (e) **Strictly increasing in $\bar{u}$** (provided utility is locally non-satiated).

Proof. Statements (a), (b), (c) holds trivially. (d) follows from the envelope theorem. Fix $\lambda \in [0, 1]$ to show (c). For any $\tilde{p}$ and u , let $x^h(\tilde{p}, u)$ be the solutions of the expenditure minimization problem. Then

$$\begin{aligned} e(\lambda p + (1 - \lambda)p', u) &= (\lambda p + (1 - \lambda)p') \cdot x^*(\lambda p + (1 - \lambda)p', u) \\ &\geq \lambda(p \cdot x^*(p, u)) + (1 - \lambda)(p' \cdot x^*(p', u)) \\ &= \lambda e(p, u) + (1 - \lambda)e(p', u). \end{aligned}$$

$\square$

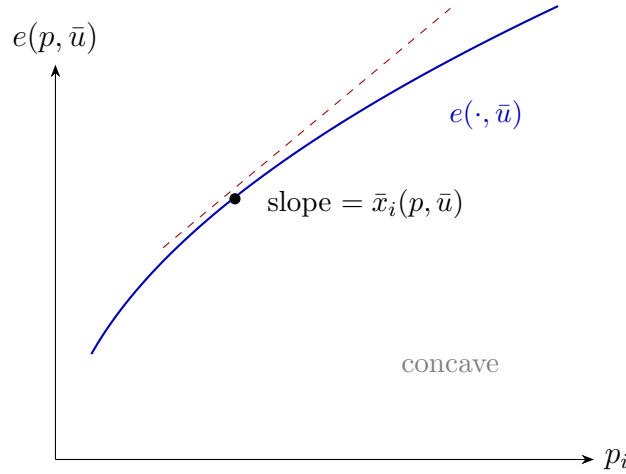


Figure 3: The expenditure function is concave in prices. The slope at any point equals the compensated demand (Shephard's lemma).

The expenditure function is the key tool for measuring welfare gains and losses from price changes.

Example 5 (Trans-Pacific Partnership (TPP)). *The U.S. is considering joining the TPP, which would change the prices faced by domestic farmers from p to p' . A farmer currently at utility level u would be harmed if prices change adversely. The government wishes to compensate the farmer just enough to restore utility u . The required subsidy is exactly $e(p', u) - e(p, u)$, the change in the expenditure function.*

10 Income Effect and Substitution Effect

We study the mechanism by which demand for a good changes in response to a change in its own price. Throughout, we work in an n -good economy with price vector $p = (p_1, \dots, p_n) \in \mathbb{R}_{++}^n$ and income $I > 0$. The consumer has a continuous, strictly quasi-concave, and locally nonsatiated utility function $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$.

For this purpose, we define an *indirect utility function* as follows:

Definition 10.1 (Indirect utility function). The **indirect utility function** is the value function of the UMP:

$$v(p, I) \equiv \max_{x \geq 0} u(x) \quad \text{subject to} \quad p \cdot x \leq I.$$

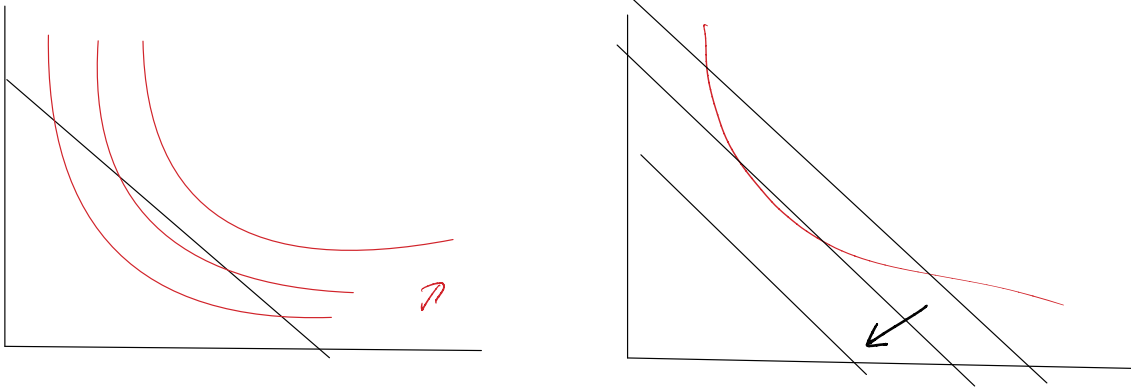
It records the maximized utility level attainable at prices $p \in \mathbb{R}_{++}^L$ and income $I > 0$.

10.1 Duality of the Consumer's Problem

Under our maintained assumptions on u , the following duality relations hold.

Proposition 9 (Duality). *Let x^* be the optimal bundle at (p, I) , and let $\bar{u} = v(p, I)$. Then:*

1. $x(p, I) = x^* = x^h(p, \bar{u})$ (Marshallian = Hicksian at the optimum);
2. $e(p, v(p, I)) = I$ (expenditure recovers income);
3. $v(p, e(p, \bar{u})) = \bar{u}$ (indirect utility recovers the target utility).



Remark 14. Whether we frame the consumer's problem as "maximize utility given a budget" or "minimize cost to achieve a target utility," we arrive at the same optimal bundle. This duality is the engine behind the Slutsky decomposition.

10.2 Changes in Real Income Induced by Price Changes

Even when nominal income I is held fixed, a change in prices alters the consumer's real income—i.e., the set of affordable bundles. We make this precise via the expenditure function.

Proposition 10 (Shephard's Lemma). *Suppose the expenditure function $e(p, \bar{u})$ is differentiable in p . Then, for each good i ,*

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}).$$

Proof. This follows from the envelope theorem applied to the EMP. The Lagrangian is $\mathcal{L} = p \cdot x - \lambda(u(x) - \bar{u})$. By the envelope theorem,

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} \Big|_{x=x^h} = x_i^h(p, \bar{u}). \quad \square$$

Using the duality relation $e(p, \bar{u}) = I$ at the optimum, we write $I(p, \bar{u})e(p, \bar{u})$. Shephard's Lemma combined with duality (Proposition 9) yields:

$$\frac{\partial I(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}) = x_i^*,$$

where x_i^* is the quantity of good i consumed at the optimum. To a first-order approximation, the change in income required to maintain utility $\bar{u}$ after a price change Δp_i is:

$$\boxed{\Delta I \approx x_i^* \cdot \Delta p_i.} \quad (4)$$

Example 6 (Water Consumption). *Suppose a consumer purchases $x_i^* = 4$ bottles of water per week. If the per-bottle price increases by $\Delta p_i = \$1$, the consumer needs approximately $\Delta I = 4 \times 1 = \4 of additional income to maintain the original utility level.*

Note that with an additional \$4 in income, the consumer could in principle re-optimize and potentially achieve a strictly higher utility level (since \$4 is a sufficient but not necessary compensation at the new prices). (4) shows locally this effect is small.

10.3 The Slutsky Equation

We now derive the fundamental decomposition of the price effect into substitution and income components.

Theorem 4 (Slutsky Equation). *Let $\bar{u} = v(p, I)$. Then for each good i and price p_i ,*

$$\boxed{\frac{\partial x_i(p, I)}{\partial p_i} = \underbrace{\frac{\partial x_i^h(p, \bar{u})}{\partial p_i}}_{\text{Substitution Effect (SE)}} - \underbrace{\frac{\partial x_i(p, I)}{\partial I} x_i^*}_{\text{Income Effect (IE)}}.} \quad (5)$$

Proof. By duality (Proposition 9), for all p and corresponding $\bar{u} = v(p, I)$:

$$x_i^h(p, \bar{u}) = x_i(p, e(p, \bar{u})).$$

Differentiating both sides with respect to p_i (applying the chain rule on the right-hand side):

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, e(p, \bar{u}))}{\partial p_i} + \frac{\partial x_i(p, e(p, \bar{u}))}{\partial I} \cdot \frac{\partial e(p, \bar{u})}{\partial p_i}. \quad (6)$$

By Shephard's Lemma (Proposition 10),

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}) = x_i^*.$$

Substituting into (6) and noting $e(p, \bar{u}) = I$:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, I)}{\partial p_i} + \frac{\partial x_i(p, I)}{\partial I} x_i^*.$$

Rearranging yields the Slutsky equation (5). □

Interpretation of the Slutsky Equation

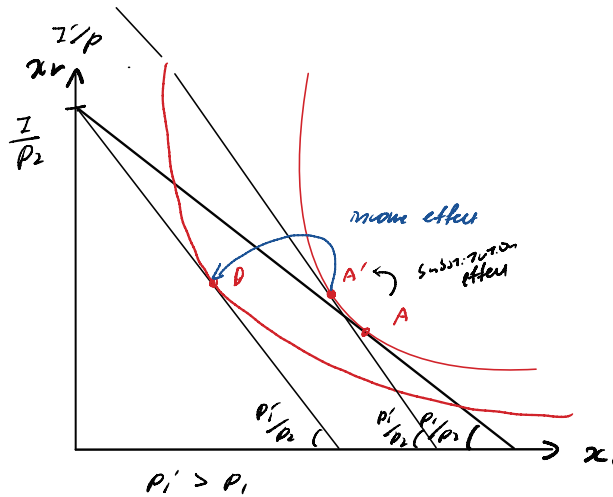
The total effect of an own-price increase on the Marshallian demand decomposes as:

$$\underbrace{\frac{\partial x_i(p, I)}{\partial p_i}}_{\text{Total Effect}} = \underbrace{\frac{\partial x_i^h(p, \bar{u})}{\partial p_i}}_{\text{SE: } \leq 0} - \underbrace{\frac{\partial x_i(p, I)}{\partial I} \cdot x_i^*}_{\text{IE}}.$$

- **Substitution Effect (SE):** The change in demand holding utility constant (movement along the indifference curve). By the negativity of self-substitution effect, $\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} \leq 0$ always. That is, compensated demand curves always slope downward.
- **Income Effect (IE):** The change in demand due to the reduction in real income caused by the price increase, multiplied by the original quantity consumed.
 - If good i is a normal good ($\frac{\partial x_i}{\partial I} \geq 0$): The IE is negative (reinforces the SE).
 - If good i is an inferior good ($\frac{\partial x_i}{\partial I} < 0$): The IE is positive (offsets the SE).
 - If good i is a Giffen good: The IE dominates the SE, so $\frac{\partial x_i}{\partial p_i} > 0$.

Remark 15. The general (cross-price) Slutsky equation for the effect of p_j on x_i takes the form

$$\frac{\partial x_i(p, I)}{\partial p_j} = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} - \frac{\partial x_i(p, I)}{\partial I} x_j^*.$$



Remark 16 (Giffen goods: an intuitive “poor household staple” story). A Giffen good is a good whose Marshallian demand increases when its own price increases:

$$\frac{\partial x_1(p, w)}{\partial p_1} > 0.$$

The standard mechanism is a very large negative income effect for a staple that is strongly inferior and takes a large budget share.

A canonical example (with supportive empirical evidence in very poor populations) is a basic staple such as rice. When the price of rice rises, the household’s real income falls: at the old consumption bundle they can no longer afford the same diet, so they become effectively poorer. For a sufficiently poor household, “better” foods (meat, eggs, vegetables, etc.) are the marginal items that get cut first. To maintain calories, the household substitutes away from these higher-quality foods and ends up relying more on the cheap staple. In that case the income effect of the rice price increase raises rice consumption so much that it outweighs the substitution effect (which always pushes rice consumption down holding utility fixed). The net result is the Giffen pattern: higher rice prices can lead to higher rice purchases.

11 Price Elasticity of Demand

Having established how demand responds to prices, we now ask the quantitative question: by how much does demand change?

In this question, we ask quantitative question: How much demand changes with respect to the change of prices?

11.1 Why Elasticity? A Unit-Free Measure of Responsiveness

A natural first measure of how demand responds to price is $\Delta x / \Delta p$. But this ratio depends on the units chosen: if we measure beer in millilitres rather than litres, the ratio changes by a factor of 1000. To obtain a unit-free measure, we express both changes as percentages. Since $\Delta x / x$ is invariant to rescaling (a factor of 1000 in numerator and denominator cancels), the price elasticity of demand is defined as the percentage change in quantity per one-percent change in price:

Price and income elasticities

$$\varepsilon = \frac{\Delta x / x}{\Delta p / p}, \quad \eta = \frac{\Delta x / x}{\Delta I / I}.$$

Both measures are dimensionless and therefore comparable across goods and markets.

11.2 Elasticity: Definition and Motivation

Definition 11.1 (Price Elasticity of Demand). The **(own-price) elasticity of demand** for good i is

$$\varepsilon_{ii} \equiv -\frac{\partial x_i / x_i}{\partial p_i / p_i} = -\frac{\partial x_i}{\partial p_i} \cdot \frac{p_i}{x_i}. \quad (7)$$

The negative sign is a convention so that $\varepsilon_{ii} \geq 0$ for ordinary (non-Giffen) goods.

Remark 17 (Unit-Free Measurement). *A naive measure of price responsiveness, $\Delta x_i / \Delta p_i$, depends on the units chosen for quantity and price. For instance, measuring beer in liters versus milliliters changes the ratio by a factor of 1000. The percentage formulation*

$$\varepsilon_{ii} = -\frac{\Delta x_i / x_i}{\Delta p_i / p_i}$$

is unit-free, since

$$\frac{\Delta x_i / x_i}{\Delta p_i / p_i} = \frac{(1000 \Delta x_i) / (1000 x_i)}{\Delta p_i / p_i},$$

and the scaling factor cancels.

Definition 11.2 (Income Elasticity of Demand). The **income elasticity of demand** for good i is

$$\eta_i \equiv \frac{\partial x_i}{\partial I} \cdot \frac{I}{x_i}.$$

Good i is normal if $\eta_i > 0$; inferior if $\eta_i < 0$; a luxury good if $\eta_i > 1$; and a necessity if $0 < \eta_i < 1$.

11.3 Application: How Revenue Changes with Prices

Consider a firm producing good i with revenue $R = p_i x_i(p, I)$. Differentiating with respect to p_i :

$$\frac{\partial R}{\partial p_i} = x_i + p_i \frac{\partial x_i}{\partial p_i} = x_i \left(1 + \frac{p_i}{x_i} \frac{\partial x_i}{\partial p_i} \right) = x_i (1 - \varepsilon_{ii}). \quad (8)$$

Revenue and Elasticity

$$\frac{\partial R}{\partial p_i} = x_i (1 - \varepsilon_{ii}).$$

- $\varepsilon_{ii} < 1$ (inelastic): A price increase raises revenue.
- $\varepsilon_{ii} = 1$ (unit elastic): Revenue is maximized; a marginal price change has no first-order revenue effect.
- $\varepsilon_{ii} > 1$ (elastic): A price increase lowers revenue.

11.4 Determinants of Price Elasticity: An Elasticity Form of the Slutsky Equation

We now connect the Slutsky equation (Theorem 4) to elasticities, providing a structural account of what makes demand more or less elastic.

Multiply both sides of the Slutsky equation (5) by $-p_i/x_i$:

$$\underbrace{-\frac{\partial x_i(p, I)}{\partial p_i} \frac{p_i}{x_i}}_{\varepsilon_{ii}} = \underbrace{-\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} \frac{p_i}{x_i}}_{\varepsilon_{ii}^h} + \underbrace{\frac{\partial x_i(p, I)}{\partial I} \frac{I}{x_i}}_{\eta_i} \cdot \underbrace{\frac{p_i x_i}{I}}_{w_i}, \quad (9)$$

where:

- ε_{ii} is the **Marshallian price elasticity**;
- ε_{ii}^h is the **compensated (Hicksian) price elasticity**—the elasticity along the indifference curve;
- η_i is the **income elasticity of demand**;
- $w_i = p_i x_i / I$ is the **expenditure (budget) share** of good i .

Determinants of Price Elasticity

The Marshallian price elasticity ε_{ii} is large when:

- (1) **Close substitutes are available** (ε_{ii}^h is large): When the consumer can easily switch to alternative goods, the compensated demand is highly responsive to price changes.
- (2) **Income elasticity is high** (η_i is large): Luxury goods ($\eta_i > 1$) tend to have larger overall elasticities because the income effect amplifies the substitution effect.
- (3) **Budget share is high** (w_i is large): Goods that command a large fraction of the consumer's budget generate a substantial real-income reduction when their price rises, magnifying the income effect.

Example 7 (Empirical Price Elasticities in the United States). *The following table reports estimated own-price elasticities for selected goods, ordered from least to most elastic.*

<i>Good</i>	$\hat{\varepsilon}_{ii}$	<i>Category</i>	<i>Key driver</i>
<i>Eggs</i>	0.1	<i>Inelastic</i>	<i>Few substitutes, small w_i</i>
<i>Beef</i>	0.4	<i>Inelastic</i>	<i>Some substitutes (poultry, pork)</i>
<i>Gasoline</i>	0.5	<i>Inelastic</i>	<i>Necessity, limited short-run substitutes</i>
<i>Housing</i>	1.2	<i>Elastic</i>	<i>Large w_i, some η_i</i>
<i>Restaurant meals</i>	2.3	<i>Elastic</i>	<i>Many substitutes (home cooking), luxury</i>
<i>Airline travel</i>	2.4	<i>Elastic</i>	<i>Substitutes (car, train), high η_i</i>
<i>Foreign travel</i>	4.1	<i>Elastic</i>	<i>High η_i (luxury), many substitutes</i>

The pattern is consistent with the three determinants in equation (9):

- **Eggs** ($\varepsilon = 0.1$): Eggs are a dietary staple with no close substitutes. The budget share w_i is negligible, so the income effect is tiny. Almost all of the (small) response is a substitution effect.
- **Gasoline** ($\varepsilon = 0.5$): In the short run, consumers cannot easily substitute away from gasoline (commute distances are fixed). The budget share is non-negligible, contributing some income effect.
- **Housing** ($\varepsilon = 1.2$): The large budget share (w_i often 25–35%) generates a substantial income effect, pushing elasticity above unity even though substitution options are limited.
- **Foreign travel** ($\varepsilon = 4.1$): This is a luxury good ($\eta_i \gg 1$) with many substitutes (domestic travel, staycations). All three factors—high ε_{ii}^h , high η_i , and moderate w_i —contribute to the large elasticity.

Goods	Is there close substitute?	luxurious goods?	high expenditure share ?
Eggs	Probably No	No	No
Beef	Yes	No	No
Gasoline	No	No	Yes/No
Housing	No	No	Yes
Restaurant meals	Yes	Yes/No	Yes
Airline travel	No	No	Yes
Foreign travel	Yes (e.g., domestic travel)	Yes	Yes

12 The Weak Axiom of Revealed Preference (MWG §2.F)

Section 12 (this section) applies the abstract WARP framework of §1.C to the Walrasian demand function $x(p, w)$. WARP in this context has sharp quantitative implications: it is equivalent to a compensated law of demand and implies that the Slutsky substitution matrix is negative semidefinite.

Throughout this section, we assume that $x(p, w)$ satisfies Walras' law: $p \cdot x(p, w) = w$ for all $(p, w) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}$.

12.1 Definition of WARP for Walrasian Demand

Definition 12.1 (WARP for Walrasian demand (MWG Def. 2.F.1)). The demand function $x(p, w)$ satisfies the weak axiom of revealed preference (WARP) if: for any two price–wealth pairs (p, w) and (p', w') ,

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \implies p' \cdot x(p, w) > w'. \quad (10)$$

Economic intuition for WARP

- The bundle $x(p, w)$ is the consumer's choice at budget (p, w) .
- If $x(p', w')$ was affordable at (p, w) (i.e. $p \cdot x(p', w') \leq w$) but was not chosen (the consumer chose the different bundle $x(p, w)$ instead), then $x(p', w')$ is revealed worse than $x(p, w)$.
- WARP then requires that $x(p, w)$ is not affordable at (p', w') (i.e. $p' \cdot x(p, w) > w'$), since otherwise the consumer would be “choosing $x(p', w')$ over $x(p, w)$ even though $x(p, w)$ was available,” contradicting the previous revealed preference.

Remark 18 (Equivalent formulation). *WARP can be restated symmetrically: if $x(p, w)$ is chosen when $x(p', w')$ was affordable, then we cannot have $x(p', w')$ chosen when $x(p, w)$ is affordable and distinct. In the language of §1.C, the demand function $x(\cdot, \cdot)$ satisfies the WARP for the family $\mathcal{B}$ of Walrasian budget sets.*

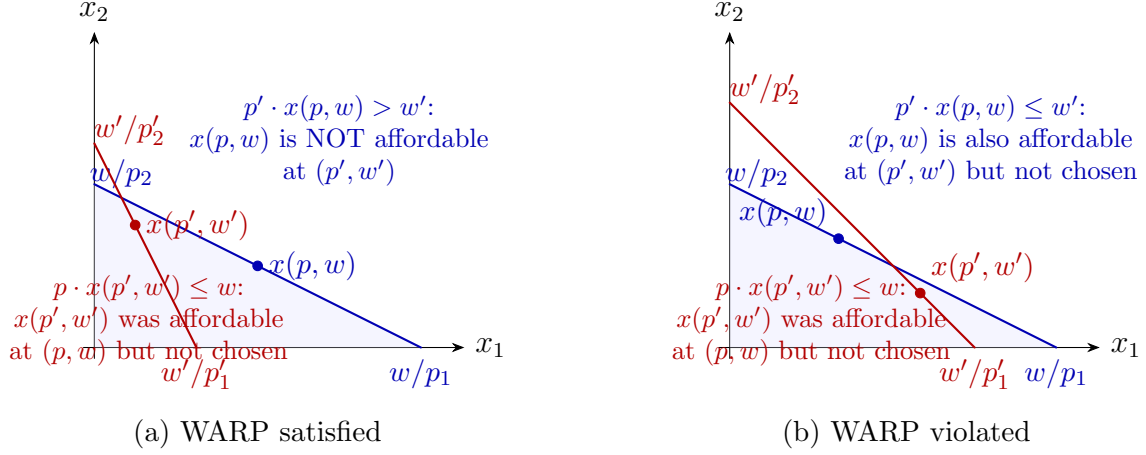


Figure 4: Panel (a): WARP satisfied. $x(p', w')$ (red) lies inside the budget set at (p, w) but was not chosen, revealing $x(p, w) \succsim x(p', w')$. WARP requires $x(p, w)$ to be outside the budget set at (p', w') , i.e. $p' \cdot x(p, w) > w'$, which holds. Panel (b): WARP violated. Both bundles lie inside each other's budget sets: $x(p', w')$ was affordable at (p, w) and $x(p, w)$ was affordable at (p', w') , so the consumer simultaneously reveals $x(p, w) \succsim x(p', w')$ and $x(p', w') \succsim x(p, w)$ with distinct bundles — a contradiction.

12.2 The Compensated Law of Demand

The central testable consequence of WARP is a generalization of the law of demand to compensated price changes.

Definition 12.2 (Slutsky compensation). A price change from p to p' is Slutsky-compensated if wealth is adjusted so that the original demand bundle remains just affordable:

$$w' = p' \cdot x(p, w).$$

Under this compensation, the consumer can still purchase the original bundle $x(p, w)$, so any change in demand reflects pure substitution.

Proposition 11 (Compensated law of demand (MWG Prop. 2.F.1)). *Suppose $x(p, w)$ satisfies Walras' law and WARP. Let p' be any price vector and set $w' = p' \cdot x(p, w)$ (Slutsky compensation). Then*

$$(p' - p) \cdot (x(p', w') - x(p, w)) \leq 0. \quad (11)$$

Moreover, if $x(p', w') \neq x(p, w)$, the inequality is strict.

Economic content of the compensated law of demand

Let $\Delta p = p' - p$ and $\Delta x = x(p', w') - x(p, w)$. The inequality $\Delta p \cdot \Delta x \leq 0$ says: price changes and compensated quantity changes move in opposite directions (on average). In particular, if only the price of good i changes ($p'_j = p_j$ for $j \neq i$) and wealth is Slutsky-compensated, then

$$(p'_i - p_i)(x_i(p', w') - x_i(p, w)) \leq 0,$$

i.e. a rise in p_i leads to a fall (or no change) in compensated demand for good i . This generalises Theorem 3 from Hicksian demand to the WARP setting.

Proof.

Case 1: $x(p', w') = x(p, w)$. Then $\Delta x = 0$ and $(p' - p) \cdot \Delta x = 0 \leq 0$.

Case 2: $x(p', w') \neq x(p, w)$.

Step 1 (Verify WARP applies). By assumption $w' = p' \cdot x(p, w)$, so $x(p, w)$ is affordable at (p', w') :

$$p' \cdot x(p, w) = w'.$$

Since $x(p', w') \neq x(p, w)$, applying WARP (10) with the roles of (p, w) and (p', w') swapped: $p' \cdot x(p, w) = w' \leq w'$ and $x(p, w) \neq x(p', w')$ together imply

$$p \cdot x(p', w') > w.$$

Step 2 (Compute $p' \cdot \Delta x$). By Walras' law $p' \cdot x(p', w') = w'$, and by definition of Slutsky compensation $w' = p' \cdot x(p, w)$. Therefore:

$$p' \cdot (x(p', w') - x(p, w)) = w' - w' = 0.$$

Step 3 (Compute $p \cdot \Delta x$). From Step 1: $p \cdot x(p', w') > w$. By Walras' law: $p \cdot x(p, w) = w$. Therefore:

$$p \cdot (x(p', w') - x(p, w)) = p \cdot x(p', w') - w > 0.$$

Step 4 (Combine).

$$(p' - p) \cdot (x(p', w') - x(p, w)) = \underbrace{p' \cdot \Delta x}_{=0} - \underbrace{p \cdot \Delta x}_{>0} < 0. \quad \square$$

12.3 Implications for the Slutsky Matrix

Definition 12.3. The *Slutsky matrix* $S(p, w)$ is defined as

$$S(p, w) = D_p x(p, w) + D_w x(p, w) \cdot x(p, w)^\top.$$

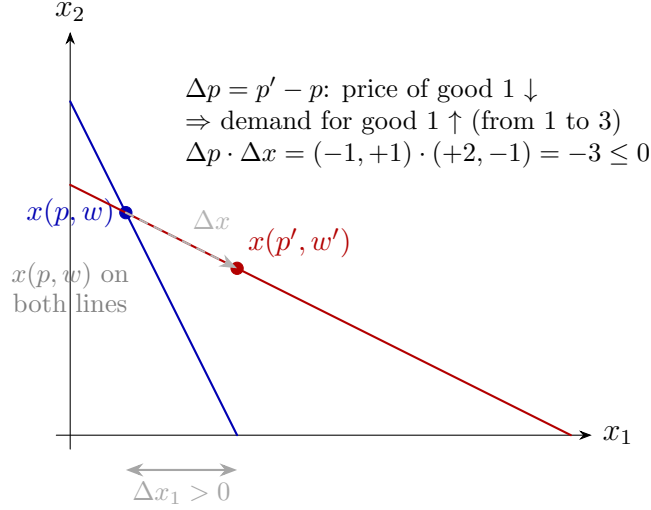


Figure 5: Compensated law of demand. The original budget line (blue line) with (p, w) is steeper (good 1 is expensive relative to good 2). After a Slutsky-compensated price change to (p', w') (good 1 becomes cheaper, $p'_1 < p_1$), the new budget line (red line) with (p, w) passes through $x(p, w)$ and is shallower. Demand shifts from $x(p, w)$ to $x(p', w')$: good 1 demand rises. The inner product $\Delta p \cdot \Delta x < 0$ confirms the compensated law of demand.

Written out explicitly:

$$S(p, w) = \begin{pmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \cdots & \frac{\partial x_1}{\partial p_L} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \cdots & \frac{\partial x_2}{\partial p_L} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_L}{\partial p_1} & \frac{\partial x_L}{\partial p_2} & \cdots & \frac{\partial x_L}{\partial p_L} \end{pmatrix} + \begin{pmatrix} \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial w} \\ \vdots \\ \frac{\partial x_L}{\partial w} \end{pmatrix} (x_1 \quad x_2 \quad \cdots \quad x_L).$$

The generic (i, j) entry is therefore

$$s_{ij}(p, w) = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, w).$$

By the Slutsky equation, this equals the Hicksian cross-price derivative:

$$s_{ij}(p, w) = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j},$$

where $\bar{u} = v(p, w)$ is the indirect utility at (p, w) .

Proposition 12 (WARP implies negative semidefinite Slutsky matrix). *Suppose $x(p, w)$ is differentiable, satisfies Walras' law and WARP. Then the Slutsky substitution matrix $S(p, w)$ is negative semidefinite: $v^\top S(p, w) v \leq 0$ for all $v \in \mathbb{R}^L$.*

Proof sketch. Consider a Slutsky-compensated price change $p' = p + tv$ (for small $t > 0$) with $w' = p' \cdot x(p, w)$.

By first-order Taylor expansion,

$$x(p', w') - x(p, w) \approx [D_p x(p, w) + D_w x(p, w) x(p, w)^\top] tv = S(p, w) tv.$$

For the details see the footnote below.⁴ Substituting into the compensated law of demand (11):

$$(p' - p) \cdot (x(p', w') - x(p, w)) \approx (tv)^\top S(p, w)(tv) = t^2 v^\top S(p, w) v \leq 0.$$

Since $t^2 > 0$, we conclude $v^\top S(p, w) v \leq 0$. □

Remark 19 ($S(p, w)p = 0$). *By differentiating Walras' law $p \cdot x(p, w) = w$ with respect to p (holding w fixed) and with respect to w (holding p fixed), one obtains:*

$$S(p, w)p = \mathbf{0}.$$

Thus prices always lie in the null space of S ; the matrix S has rank at most $L - 1$.

⁴We want to expand $x(p', w')$ around (p, w) where $p' = p + tv$ and $w' = p' \cdot x(p, w) = w + tv \cdot x(p, w)$ (Slutsky compensation). Write $\Delta p = tv$ and $\Delta w = w' - w = tv \cdot x(p, w)$. By differentiability of x in (p, w) :

$$\begin{aligned} x(p', w') &= x(p + tv, w + tv \cdot x(p, w)) \\ &= x(p, w) + D_p x(p, w) \cdot (tv) + D_w x(p, w) \cdot (tv \cdot x(p, w)) + o(t) \\ &= x(p, w) + t \left[D_p x(p, w) \cdot v + D_w x(p, w) (v \cdot x(p, w)) \right] + o(t). \end{aligned}$$

Now note that $v \cdot x(p, w) = x(p, w)^\top v$, so the term in brackets is

$$D_p x(p, w) v + D_w x(p, w) x(p, w)^\top v = \left[D_p x(p, w) + D_w x(p, w) x(p, w)^\top \right] v = S(p, w) v.$$

That's exactly the definition of the Slutsky matrix from Definition 12.3. So:

$$x(p', w') - x(p, w) = t S(p, w) v + o(t).$$

12.4 WARP versus Full Rationality

WARP and utility maximization (MWG Prop. 2.F.2–2.F.3)

- **Rationality $\Rightarrow$ WARP.** If $x(p, w)$ is derived from utility maximization over a rational preference relation $\succsim$, then $x(p, w)$ satisfies WARP. (This follows because $\succsim$ rationalizes the induced choice rule, which satisfies WARP by Proposition 1 on the choice function.)
- **WARP $\nRightarrow$ Rationality (when $L \geq 3$).** For $L \geq 3$ goods, WARP is strictly weaker than full rationality: there exist demand functions satisfying WARP that cannot be rationalized by any utility function. Intuitively, WARP only restricts pairwise choices, while rationality requires consistency over all possible budget sets simultaneously.
- **$L = 2$ special case.** When $L = 2$, WARP (together with Walras' law and homogeneity of degree zero) is equivalent to rationalizability: any demand function satisfying these conditions can be derived from a preference relation.
- **Full rationality implies symmetry of S .** Utility maximization implies that $S(p, w)$ is not only negative semidefinite but also symmetric: $S_{ij} = S_{ji}$. WARP alone does not imply symmetry of S when $L \geq 3$; symmetry is the additional restriction imposed by rationality beyond WARP.

Remark 20 (Revealed-preference tests in empirical work). *The compensated law of demand (11) is directly testable from observed demand data: given two price–quantity observations (p, x) and (p', x') , WARP requires that $(p' - p) \cdot (x' - x) \leq 0$ whenever $p \cdot x' \leq p' \cdot x$ (i.e. x' was affordable at (p, w)) and $x' \neq x$. More demanding tests based on full rationality are provided by the Strong Axiom of Revealed Preference (SARP) and the Generalised Axiom of Revealed Preference (GARP). See separate note on the relationship among SARP, WARP, and GARP. Different authors use different names. Afriat condition is equivalent with GARP in Echenique and Chambers's book, which seems equivalent to SARP in MWG.*

Remark 21. *The Kihlstrom–Mas-Colell–Sonnenschein (1976) paper (henceforth KMS) provides the definitive characterisation of the set of demand functions satisfying the Weak Axiom of Revealed Preference (WARP). They study the relationship between WARP and NSD of the Slutsky matrix.*

13 Testable Implications of Utility maximization (MWG §3.E, 3.G)

The demand function $x^*(p, I)$ arises from utility maximization. This origin imposes observable, testable restrictions on the demand function. We have already established two in Section 7.1:

- **Homogeneity of degree zero** (Proposition 6): $x^*(\lambda p, \lambda I) = x^*(p, I)$ for all $\lambda > 0$.
- **Walras' law** (Proposition 7): $p \cdot x^*(p, I) = I$ whenever u is locally nonsatiated.

Two further restrictions emerge from the duality structure of the UMP.

13.1 The Indirect Utility Function and Roy's Identity

Remember that the **indirect utility function** is the value function of the UMP:

$$v(p, I) \equiv \max_{x \geq 0} u(x) \quad \text{subject to} \quad p \cdot x \leq I.$$

It records the maximized utility level attainable at prices $p \in \mathbb{R}_{++}^L$ and income $I > 0$.

The indirect utility function inherits several properties from the structure of the UMP.

Proposition 13 (Properties of $v(p, I)$). *Suppose u is continuous and locally nonsatiated. Then $v(p, I)$ is:*

1. Homogeneous of degree zero in (p, I) : $v(\lambda p, \lambda I) = v(p, I)$ for all $\lambda > 0$.
2. Nondecreasing in I and nonincreasing in each p_i .
3. Quasiconvex in (p, I) .
4. Continuous in (p, I) .

Proof sketch. (i) Homogeneity. Multiplying all prices and income by $\lambda > 0$ leaves the budget set $\{x \geq 0 : p \cdot x \leq I\}$ unchanged, so the constrained maximum is unaffected.

(ii) Monotonicity. A higher income relaxes the constraint, so the maximum cannot decrease. A higher p_i tightens the constraint, so the maximum cannot increase.

(iii) Quasiconvexity. The set of (p, I) that make a given consumption bundle affordable is convex; taking a convex combination of two “budget-constrained” situations leaves the best attainable utility no higher than the better of the two.

(iv) Continuity. By the Maximum Theorem: the budget correspondence is continuous in (p, I) and u is continuous, so v is continuous. $\square$

The key link between the indirect utility function and Marshallian demand is **Roy's identity**, which allows one to recover the entire demand system from the single scalar function v .

Proposition 14 (Roy's Identity (MWG Prop. 3.G.4)). *Suppose $v(p, I)$ is differentiable and $\frac{\partial v}{\partial I}(p, I) \neq 0$. Then the Marshallian demand for good i satisfies*

$$x_i^*(p, I) = - \frac{\partial v(p, I) / \partial p_i}{\partial v(p, I) / \partial I}.$$

Proof. The Lagrangian for the UMP is $\mathcal{L} = u(x) - \lambda(p \cdot x - I)$, with optimal Lagrange multiplier $\lambda^* \geq 0$ (the marginal utility of income). By the **envelope theorem**, the partial derivatives of the value function v with respect to parameters equal the corresponding partial derivatives of $\mathcal{L}$ evaluated at the optimum:

$$\frac{\partial v}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} \Big|_{x=x^*} = -\lambda^* x_i^*, \quad \frac{\partial v}{\partial I} = \frac{\partial \mathcal{L}}{\partial I} \Big|_{x=x^*} = \lambda^*.$$

Dividing gives $\partial v / \partial p_i / \partial v / \partial I = -x_i^*$, which rearranges to Roy's identity. $\square$

Roy's Identity: recovering demand from the indirect utility function

Given the indirect utility function $v(p, I)$, Marshallian demands are recovered by:

$$x_i^*(p, I) = - \frac{\partial v(p, I) / \partial p_i}{\partial v(p, I) / \partial I}, \quad i = 1, \dots, L.$$

Intuition: A one-unit rise in p_i reduces utility by $\lambda^* x_i^*$ (it costs x_i^* extra units of income, each worth λ^* in utility). Dividing by λ^* converts this utility loss back into income units, yielding x_i^* . Thus the quantity demanded equals the income-equivalent utility cost of a unit price rise.

Example 8 (Roy's identity for Cobb-Douglas). For $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ with $\alpha \in (0, 1)$, the indirect utility function (substituting the optimal demands $x_1^* = \alpha I / p_1$, $x_2^* = (1 - \alpha) I / p_2$) is

$$v(p_1, p_2, I) = \left(\frac{\alpha}{p_1} \right)^\alpha \left(\frac{1 - \alpha}{p_2} \right)^{1-\alpha} I.$$

Applying Roy's identity to good 1:

$$x_1^* = - \frac{\partial v / \partial p_1}{\partial v / \partial I} = - \frac{-\alpha v / p_1}{v / I} = \frac{\alpha I}{p_1},$$

recovering the Cobb-Douglas Marshallian demand.

13.2 Symmetry of the Slutsky Matrix

We showed in Section 12 that WARP implies the Slutsky matrix is negative semidefinite (Proposition 12). Full utility maximization—rationality with complete and transitive preferences—implies the stronger result that the Slutsky matrix is also symmetric.

Recall from the Slutsky equation (Theorem 4) that the (i, j) entry of the Slutsky matrix is

$$s_{ij}(p, I) = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} = \frac{\partial x_i(p, I)}{\partial p_j} + \frac{\partial x_i(p, I)}{\partial I} x_j^*(p, I).$$

Proposition 15 (Slutsky symmetry (MWG Prop. 3.G.1)). *If $x^*(p, I)$ is derived from utility maximization by a continuous, locally nonsatiated utility function, then the Slutsky matrix $S(p, I)$ is symmetric: $s_{ij} = s_{ji}$ for all i, j .*

Proof. By Shephard's Lemma (Proposition 10), Hicksian demand is the gradient of the expenditure function:

$$x_i^h(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_i}.$$

Therefore

$$s_{ij} = \frac{\partial x_i^h}{\partial p_j} = \frac{\partial^2 e}{\partial p_j \partial p_i}.$$

By **Young's theorem** (equality of mixed partial derivatives), since e is twice continuously differentiable:

$$\frac{\partial^2 e}{\partial p_j \partial p_i} = \frac{\partial^2 e}{\partial p_i \partial p_j} = s_{ji}.$$

Hence S is symmetric. □

Slutsky symmetry: a cross-equation restriction on demand

Utility maximization imposes, for all pairs $i \neq j$:

$$\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial I} x_j^* = \frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial I} x_i^*.$$

This is a testable restriction on the Marshallian demand system. It follows purely from the fact that both x_i^h and x_j^h are partial derivatives of the same scalar function $e(p, \bar{u})$, so they must satisfy the cross-derivative condition (Young's theorem). With L goods there are $\frac{L(L-1)}{2}$ such restrictions.

Remark 22 (WARP vs. full rationality). *For $L = 2$ goods, NSD of the Slutsky matrix already forces symmetry (a 2×2 matrix satisfying $p^\top S = 0$ and $S \preceq 0$ must be symmetric). For $L \geq 3$ goods, symmetry is an additional restriction beyond WARP: the Slutsky matrix can be NSD but not symmetric. The Strong Axiom of Revealed Preference (SARP), which is equivalent to full rationality, is needed to obtain symmetry for $L \geq 3$.*

13.3 The Integrability Theorem

The four properties established above—homogeneity, Walras' law, Slutsky symmetry, and Slutsky NSD—are not only necessary for utility maximization: they are also sufficient.

Theorem 5 (Integrability Theorem (MWG Prop. 3.H.1)). *A continuously differentiable demand function $x(p, I)$ on $\mathbb{R}_{++}^L \times \mathbb{R}_{++}$ is rationalised by a continuous, locally nonsatiated utility function if and only if it satisfies:*

1. **Homogeneity of degree zero:** $x(\lambda p, \lambda I) = x(p, I)$ for all $\lambda > 0$.
2. **Walras' law:** $p \cdot x(p, I) = I$ for all (p, I) .
3. **Slutsky symmetry:** The Slutsky matrix $S(p, I)$ is symmetric.
4. **Slutsky NSD:** $z^\top S(p, I) z \leq 0$ for all $z \in \mathbb{R}^L$.

Proof sketch. **Necessity** (utility maximization $\Rightarrow$ all four conditions):

- (i) and (ii): Propositions 6 and 7.
- (iii) Symmetry: Proposition 15 via Young's theorem.
- (iv) NSD: By the compensated law of demand (Proposition 11 and 12), $dp \cdot dx^h \leq 0$ for any compensated price change, i.e. $dp^\top S dp \leq 0$ for all dp .

Sufficiency (all four conditions $\Rightarrow$ rationalisability): Given $x(p, I)$ satisfying (i)–(iv), construct the expenditure function by integrating:

$$e(p, \bar{u}) = \int_{p^0}^p x^h(q, \bar{u}) \cdot dq,$$

where x^h is defined implicitly via the Slutsky equation. This is a line integral of a vector field. See appendix for the detail.

- Symmetry (iii) is the integrability condition. By the Frobenius theorem, the PDE system $\partial e / \partial p_i = x_i^h(p)$ has a (path-independent) solution if and only if $\partial h_i / \partial p_j = \partial h_j / \partial p_i$ —which is exactly Slutsky symmetry. Hence e is well-defined. See the remark below.
- NSD (iv) ensures concavity. Since $\partial^2 e / \partial p_i \partial p_j = s_{ij}$ and $S \preceq 0$, the constructed e is concave in p , as required for a valid expenditure function.
- The utility function is recovered via the duality $u(x) = \min_p \{v : e(p, v) \geq p \cdot x\}$ by duality. This says: the utility of bundle x is the highest utility level that x can “afford” at every price vector simultaneously.

□

Remark 23. (*The Construction of u : Support Function Duality*) The recovery formula rests on the **supporting hyperplane theorem** for closed convex sets.

Step 1: Upper contour sets and the expenditure function. For each utility level v , the upper contour set is

$$U(v) = \{x \in \mathbb{R}_+^N : u(x) \geq v\},$$

which is convex (by quasi-concavity of u) and closed (by continuity of u). The expenditure function, for fixed v , is precisely the support function of $U(v)$:

$$e(p, v) = \min_{x \in U(v)} p \cdot x.$$

Geometrically, $e(p, v)$ is the signed distance (scaled by $\|p\|$) from the origin to the supporting hyperplane of $U(v)$ with outward normal p .

Step 2: Recovering $U(v)$ via the supporting hyperplane theorem. By the supporting hyperplane theorem, any closed convex set equals the intersection of all half-spaces that contain it. Applied to $U(v)$:

$$U(v) = \{x \in \mathbb{R}_+^N : p \cdot x \geq e(p, v) \ \forall p \gg 0\}.$$

Each price vector p defines a half-space $\{x : p \cdot x \geq e(p, v)\}$, and $U(v)$ is exactly their intersection.

Step 3: Recovering $u(x)$. From Step 2, $x \in U(v)$ —equivalently, $u(x) \geq v$ —if and only if $p \cdot x \geq e(p, v)$ for all $p \gg 0$. Taking the supremum over all such v yields:

$$u(x) = \sup \{v : e(p, v) \leq p \cdot x \ \forall p \gg 0\}.$$

Since e is continuous and strictly increasing in v (by local nonsatiation), the supremum is attained and we may write $\max$ in place of $\sup$.

The four testable conditions for rationalisability

A demand system $x(p, I)$ is consistent with utility maximization if and only if:

- | | | |
|-------|------------------|---|
| (i) | Homogeneity | $x(\lambda p, \lambda I) = x(p, I)$ for all $\lambda > 0$ |
| (ii) | Walras' law | $p \cdot x(p, I) = I$ |
| (iii) | Slutsky symmetry | $s_{ij} = s_{ji}$ for all $i \neq j$ |
| (iv) | Slutsky NSD | $z^\top S(p, I) z \leq 0$ for all $z \in \mathbb{R}^L$ |

Conditions (i) and (ii) are linear restrictions, easy to impose by construction. Condition (iii) provides $\frac{L(L-1)}{2}$ cross-equation restrictions and is the key integrability condition (it makes the expenditure function well-defined). Condition (iv) adds the curvature requirement. Together they are the complete set of restrictions imposed by consumer rationality.

Remark 24 (Empirical testing of rationality). *In applied demand analysis one estimates a parametric demand system (e.g. the Almost Ideal Demand System) and tests whether the estimated Slutsky matrix is symmetric and NSD. Rejection of symmetry is the standard test of the rational-consumer hypothesis. Rejection of NSD alone (without symmetry failure) is far less common.*

Remark 25 (Summary: the duality cycle). *The integrability theorem closes the loop of consumer duality:*

$$\underbrace{u(x)}_{\text{direct utility}} \xrightarrow{\text{UMP}} \underbrace{x^*(p, I)}_{\text{Marshallian}} \xrightarrow{\text{Slutsky}} \underbrace{x^h(p, \bar{u})}_{\text{Hicksian}} = \nabla_p \underbrace{e(p, \bar{u})}_{\text{expenditure fn.}} \xrightarrow{\text{duality}} u(x).$$

The Integrability Theorem says: any x^ satisfying conditions (i)–(iv) is a valid entry point into this cycle and comes from some well-behaved utility function.*

14 Welfare Evaluation of Price Changes

In this section we study how to measure, in monetary terms, the welfare gain or loss a consumer experiences when prices change. We develop three measures—the compensating variation (CV), the equivalent variation (EV), and the area variation (AV, also called the change in Marshallian consumer surplus)—and establish their relationship.

Throughout, we consider a consumer with a continuous, strictly quasi-concave, and locally nonsatiated utility function $u : \mathbb{R}_+^N \rightarrow \mathbb{R}$. The consumer faces an initial price vector p^0 and a new price vector p^1 , with income I held fixed. For notational simplicity, we focus on a change in the price of good 1 alone: $p_1^0 = a$ and $p_1^1 = b$ with $b > a$ (a price increase), while $p_{-1}^0 = p_{-1}^1 = \bar{p}_{-1}$.

Example 9 (Electricity price increase). *An earthquake damages a power plant, forcing an electricity company to raise the price of power from a to b . The government orders the company to compensate consumers. How much compensation is required to leave consumers no worse off than before?*

Denote by $u^0 = v(p^0, I)$ and $u^1 = v(p^1, I)$ the utility levels before and after the price change, where v is the indirect utility function. Then $u^1 < u^0$.

14.1 A Money Metric: The Indirect Utility Function

Given any price vector p and any indirect utility function $v(p, I)$, we know that v is strictly increasing in I (by local nonsatiation). Hence, for a fixed reference price vector $\bar{p}$, the function

$$e(\bar{p}, v(p, I))$$

is a money metric indirect utility function: it measures utility in dollar units at reference prices $\bar{p}$. It is strictly increasing in I and provides a valid indirect utility function representing $\succsim$.

If $v(p', I) > v(p^0, I)$, then $e(\bar{p}, v(p', I)) > e(\bar{p}, v(p^0, I))$: the consumer is better off, and the difference is expressed in dollars. Both CV and EV are instances of this idea, differing only in the choice of reference prices.

14.2 Compensating Variation

Definition 14.1 (Compensating Variation). The **compensating variation** (CV) is the amount of income the consumer must receive after the price change to restore the original utility level u^0 :

$$\text{CV} \equiv e(p^1, u^0) - e(p^0, u^0).$$

Since $e(p^0, u^0) = I$, this simplifies to

$$\text{CV} = e(p^1, u^0) - I.$$

Interpretation. The CV answers: “After the price change occurs, how much additional income would the consumer need to be just as well off as before?” Equivalently, $CV = e(p^1, u^0) - e(p^0, u^0)$, which is the difference between the cost of achieving u^0 and the cost of achieving u^1 at the new prices—both evaluated at p^1 .

For a price increase ($b > a$), the consumer is worse off, so $CV > 0$: the consumer needs a positive transfer to be compensated.

CV as area under compensated demand. By Shephard’s lemma, $\partial e(p, u^0)/\partial p_1 = x_1^h(p, u^0)$, so:

$$CV = e(p^1, u^0) - e(p^0, u^0) = \int_a^b x_1^h(p_1, \bar{p}_{-1}^0, u^0) dp_1, \quad (12)$$

where $\bar{p}_{-1}^0$ is the fixed price of the other goods (other than good 1), $p^0 = (p_1^0, \bar{p}_{-1}^0) = (a, \bar{p}_{-1}^0)$ and $p^1 = (p_1^1, \bar{p}_{-1}^0) = (b, \bar{p}_{-1}^0)$. The CV is the area to the left of the initial Hicksian (compensated) demand curve $x_1^h(\cdot, u^0)$ between $p_1^0 = a$ and $p_1^1 = b$.

14.3 Equivalent Variation

Definition 14.2 (Equivalent Variation). The **equivalent variation** (EV) is the amount of income the consumer would be willing to pay, before the price change, to avoid it:

$$EV \equiv e(p^1, u^1) - e(p^0, u^1).$$

Since $e(p^1, u^1) = I$, this simplifies to

$$EV = I - e(p^0, u^1).$$

Interpretation. The EV answers: “At the old prices, how much income would the consumer give up to prevent the price change?” Equivalently, EV is the income-equivalent of the welfare change measured at the old prices.

For a price increase, $EV > 0$: the consumer would pay a positive amount to avoid it.

EV as area under compensated demand. By Shephard’s lemma applied at utility level u^1 :

$$EV = e(p^1, u^1) - e(p^0, u^1) = \int_a^b x_1^h(p_1, \bar{p}_{-1}, u^1) dp_1. \quad (13)$$

The EV is the area to the left of the final Hicksian demand curve $x_1^h(\cdot, u^1)$ between $p_1^0 = a$ and $p_1^1 = b$.

14.4 The Area Variation (Marshallian Consumer Surplus)

A practical difficulty with both CV and EV is that the Hicksian demand functions $x_1^h(\cdot, u^0)$ and $x_1^h(\cdot, u^1)$ are not directly observable. Only the Marshallian demand $x_1(p, I)$ can be estimated from data. This motivates a third measure.

Definition 14.3 (Area Variation). The **area variation** (AV), also called the change in **Marshallian consumer surplus**, is

$$AV = \int_a^b x_1(p_1, \bar{p}_{-1}, I) dp_1. \quad (14)$$

It is the area to the left of the Marshallian (ordinary) demand curve between $p_1 = a$ and $p_1 = b$.

The AV is directly computable from observable demand data and does not require knowledge of the consumer's preferences.

Remark 26. *AV is often referred to as consumer surplus, largely for historical reasons tracing back to Marshall's terminology. Subsequent work questioned the welfare interpretation of this measure and introduced equivalent variation (EV) and compensating variation (CV) as theoretically grounded alternatives. AV is best viewed as a practical approximation to EV and CV—an interpretation formalized by Willig.*

AV may not be uniquely defined. See the appendix.

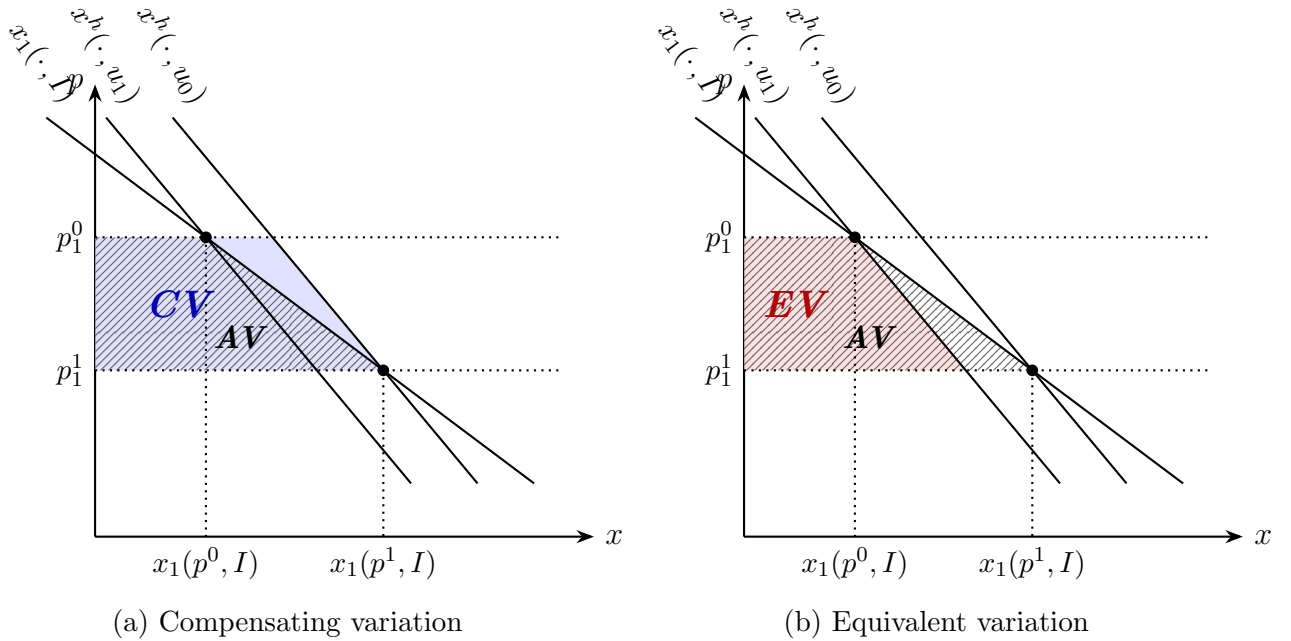


Figure 6: EV and CV from Hicksian demand curves. Price increases from p_1^0 to p_1^1 , decreasing utility from u_0 to u_1 . The two steeper lines are the Hicksian demands $x^h(\cdot, u_0)$ and $x^h(\cdot, u_1)$; the less steep line is the Marshallian demand $x_1(\cdot, I)$ as we are assuming normal goods. Panel (a): EV (blue) is the area to the left of $x^h(\cdot, u_1)$; AV (dashed) is the area to the left of $x_1(\cdot, I)$. Panel (b): CV (red) is the area to the left of $x^h(\cdot, u_0)$; AV (dashed) is the area to the left of $x_1(\cdot, I)$. For a normal good with a price increase, $CV > AV > EV$.

14.5 The Ordering: CV, AV, and EV

The three measures are related through the Slutsky equation. The key observation is that, for a price increase of a normal good, the Hicksian demand curve at utility u^0 lies to the right of the Marshallian demand curve, which in turn lies to the right of the Hicksian demand curve at u^1 .

Proposition 16 (Ordering of welfare measures for normal goods). *Suppose good 1 is a normal good ($\partial x_1 / \partial I > 0$) and $b > a$ (a price increase). Then:*

$$CV \geq AV \geq EV > 0.$$

Proof. The Slutsky equation gives, for each p_1 ,

$$\frac{\partial x_1^h(p_1, u)}{\partial p_1} = \frac{\partial x_1(p_1, I)}{\partial p_1} + \frac{\partial x_1(p_1, I)}{\partial I} x_1.$$

Since the income effect $(\partial x_1 / \partial I) x_1 > 0$ for a normal good, the Hicksian demand curve is steeper (less elastic) than the Marshallian demand curve. Moreover, since $u^0 > u^1$ (the price increase lowers utility) and good 1 is normal, we have

$$x_1^h(p_1, u^0) \geq x_1(p_1, I) \geq x_1^h(p_1, u^1) \quad \text{for all } p_1 \in [a, b].$$

Integrating from a to b yields $CV \geq AV \geq EV$. □

Remark 27 (When all three coincide). *If the utility function is quasi-linear in a numeraire good, i.e. $u(x_1, \dots, x_{N-1}, x_N) = \phi(x_1, \dots, x_{N-1}) + x_N$, then the income effect for goods $1, \dots, N-1$ is identically zero. In this case, the Hicksian and Marshallian demand curves coincide, and*

$$CV = AV = EV.$$

More generally, when the income effect is small (small budget share $w_1 = p_1 x_1 / I$), all three measures are approximately equal, and AV is a good approximation to both CV and EV.

Appendix

A Integrability Condition and Slutsky symmetry

The question is as follows:

Does there exist a scalar function $e : \mathbb{R}_{++}^L \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}) \quad \text{for all } i = 1, \dots, L?$$

In the language of multivariable calculus, this asks: is the vector field $h = (h_1, \dots, h_L)$ the gradient of some scalar function? Equivalently: is the differential 1-form

$$\omega = \sum_{i=1}^L x_i^h(p, \bar{u}) dp_i$$

exact?

A.1 Math Review: Exact Differential Forms and Conservative Vector Fields

Recall from multivariable calculus the following classical result.

Theorem 6 (Poincaré Lemma / Exactness Criterion). *Let $\Omega \subseteq \mathbb{R}^L$ be an open, simply connected domain, and let $f_1, \dots, f_L : \Omega \rightarrow \mathbb{R}$ be C^1 functions. The following are equivalent:*

(a) **Exactness.** *There exists a C^2 function $F : \Omega \rightarrow \mathbb{R}$ such that*

$$\frac{\partial F}{\partial p_i} = f_i(p) \quad \text{for all } i.$$

(b) **Path independence.** *For any two points $p^0, p^1 \in \Omega$ and any piecewise- C^1 path γ from p^0 to p^1 ,*

$$\int_{\gamma} \sum_i f_i(p) dp_i$$

depends only on the endpoints, not on the path γ .

(c) **Symmetry of the Jacobian.** *For all $i, j \in \{1, \dots, L\}$,*

$$\frac{\partial f_i}{\partial p_j} = \frac{\partial f_j}{\partial p_i}.$$

Remark 28. *In $L = 2$ dimensions, condition (c) reduces to the familiar $\partial f_1 / \partial p_2 = \partial f_2 / \partial p_1$, which is the criterion for an exact differential $f_1 dp_1 + f_2 dp_2 = dF$. In $L = 3$, it is equivalent to $\nabla \times f = 0$ (the vector field is irrotational / conservative). In general L dimensions, the result is a special case of the Frobenius theorem (or equivalently, the Poincaré lemma for 1-forms on simply connected domains).*

The logical structure is:

(a) Exactness $\implies$ (c) Symmetry

is immediate from Young's theorem ($\partial^2 F / \partial p_j \partial p_i = \partial^2 F / \partial p_i \partial p_j$). The substantive content is the converse:

(c) Symmetry $\implies$ (a) Exactness,

which says that the Jacobian symmetry condition is sufficient for a potential function to exist.

Application to Consumer Theory

Now set $f_i(p) = x_i^h(p, \bar{u})$ (Hicksian demand for good i , with $\bar{u}$ fixed) and $F(p) = e(p, \bar{u})$ (the expenditure function we wish to construct). The domain $\Omega = \mathbb{R}_{++}^L$ is open and simply connected (it is convex). The theorem then says:

The expenditure function $e(p, \bar{u})$ satisfying $\nabla_p e = h(\cdot, \bar{u})$ exists $\iff$ the Jacobian of h is symmetric:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_j} = \frac{\partial h_j(p, \bar{u})}{\partial p_i} \quad \text{for all } i, j.$$

But $\partial x_i^h / \partial p_j = s_{ij}$ is exactly the (i, j) entry of the Slutsky matrix. So the symmetry condition is:

$$s_{ij} = s_{ji} \quad \text{for all } i, j,$$

which is **Slutsky symmetry**.

A.2 Constructing e by Integration

When symmetry holds, the expenditure function is recovered by a line integral. Fix a reference price p^0 and a reference utility level $\bar{u}$ with $e(p^0, \bar{u})$ known (e.g. from observed data). For any target price p , choose any piecewise- C^1 path γ from p^0 to p in $\mathbb{R}_{++}^L$ and define:

$$e(p, \bar{u}) = e(p^0, \bar{u}) + \int_{\gamma} \sum_{i=1}^L x_i^h(q, \bar{u}) dq_i.$$

Slutsky symmetry guarantees this integral is **path-independent**, so e is well-defined regardless of which path γ is chosen.

A.3 Why Symmetry Guarantees Path Independence: A Direct Argument

We verify directly that the Jacobian symmetry condition implies path independence. Consider two paths γ_1 and γ_2 from p^0 to p , and let $\Gamma = \gamma_1 - \gamma_2$ denote the closed loop traversing γ_1 forward and γ_2 backward. We need to show:

$$\oint_{\Gamma} \sum_i x_i^h dq_i = 0.$$

By Stokes' theorem (Green's theorem in $L = 2$), this loop integral equals a surface integral of the “curl” of h . In two dimensions:

$$\oint_{\Gamma} h_1 dq_1 + h_2 dq_2 = \iint_D \left(\frac{\partial h_2}{\partial q_1} - \frac{\partial h_1}{\partial q_2} \right) dq_1 dq_2,$$

where D is the region enclosed by Γ . If Slutsky symmetry holds ($\partial h_1 / \partial q_2 = \partial h_2 / \partial q_1$), the integrand vanishes identically, so the loop integral is zero. The argument extends to

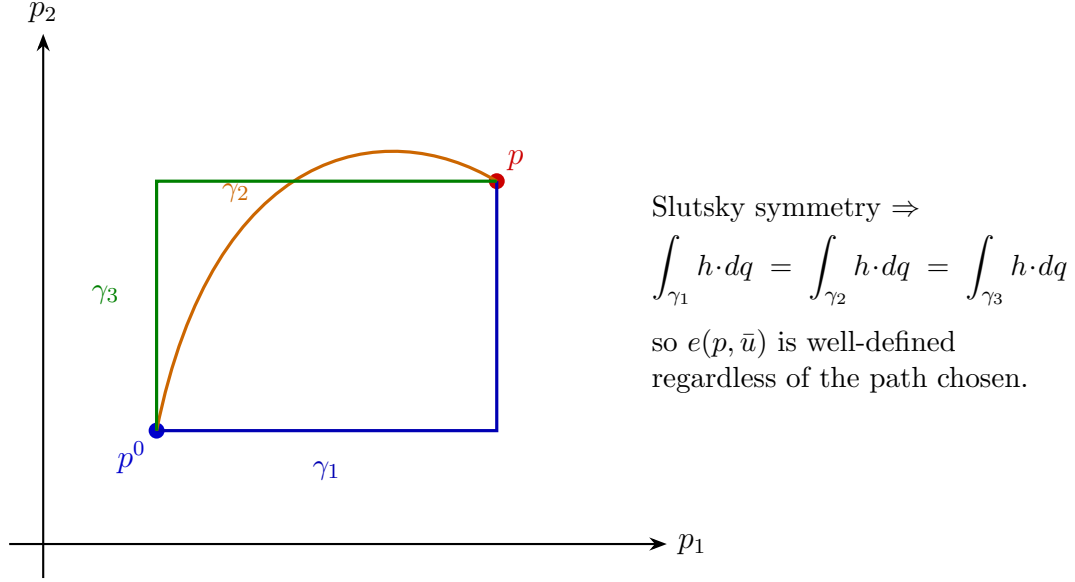


Figure 7: Path independence of the line integral. Slutsky symmetry ensures that integrating the Hicksian demand vector field $h = (h_1, \dots, h_L)$ from p^0 to p gives the same value along any path γ in $\mathbb{R}_{++}^L$.

$L \geq 3$ dimensions via the general Stokes' theorem: the exterior derivative of the 1-form $\omega = \sum_i x_i^h dp_i$ is

$$d\omega = \sum_{i < j} \left(\frac{\partial h_j}{\partial p_i} - \frac{\partial x_i^h}{\partial p_j} \right) dp_i \wedge dp_j.$$

Slutsky symmetry ($s_{ij} = s_{ji}$) makes every coefficient zero, so $d\omega = 0$, which means ω is closed. On the simply connected domain $\mathbb{R}_{++}^L$, closed $\Leftrightarrow$ exact (by the Poincaré lemma), so $\omega = de$ for some function e .

A.4 What Goes Wrong Without Symmetry

If Slutsky symmetry fails—say $s_{12} \neq s_{21}$ —then different paths of integration from p^0 to p yield different values:

$$\int_{\gamma_1} h \cdot dq \neq \int_{\gamma_2} h \cdot dq.$$

No single-valued function $e(p, \bar{u})$ can simultaneously satisfy $\nabla_p e = h$, because a gradient field must have a symmetric Jacobian (Young's theorem). In economic terms: the demand system cannot have arisen from any well-behaved utility function, because the expenditure function that would rationalise it simply does not exist.

A.5 Concrete Example: $L = 2$

Example 10. Consider a candidate Hicksian demand system:

$$h_1(p_1, p_2, \bar{u}) = \frac{\bar{u}}{p_1^{1/2} p_2^{1/2}}, \quad h_2(p_1, p_2, \bar{u}) = \frac{\bar{u}}{p_1^{1/2} p_2^{1/2}}.$$

Check symmetry:

$$\frac{\partial h_1}{\partial p_2} = -\frac{\bar{u}}{2 p_1^{1/2} p_2^{3/2}}, \quad \frac{\partial h_2}{\partial p_1} = -\frac{\bar{u}}{2 p_1^{3/2} p_2^{1/2}}.$$

These are not equal (unless $p_1 = p_2$), so $s_{12} \neq s_{21}$. No expenditure function exists—this demand system is **not rationalisable**.

Now consider the Cobb–Douglas Hicksian demands (with equal shares):

$$h_1(p_1, p_2, \bar{u}) = \bar{u} \left(\frac{p_2}{p_1} \right)^{1/2}, \quad h_2(p_1, p_2, \bar{u}) = \bar{u} \left(\frac{p_1}{p_2} \right)^{1/2}.$$

Check symmetry:

$$\frac{\partial h_1}{\partial p_2} = \frac{\bar{u}}{2 p_1^{1/2} p_2^{1/2}}, \quad \frac{\partial h_2}{\partial p_1} = \frac{\bar{u}}{2 p_1^{1/2} p_2^{1/2}}.$$

These are equal, so $s_{12} = s_{21}$. The expenditure function exists and equals $e(p, \bar{u}) = 2\bar{u} p_1^{1/2} p_2^{1/2}$, which one can verify by checking $\partial e / \partial p_i = x_i^h$.

B Path-Dependence of Consumer Surplus with Multiple Price Change

B.1 The Problem: Multi-Price Changes and Path-Dependence

When only a single price changes (say p_1 moves from a to b , all else fixed), the Marshallian consumer surplus

$$\Delta S = - \int_a^b x_1(p_1, \bar{p}_{-1}, I) dp_1$$

is an ordinary one-dimensional integral and is unambiguous.

Auerbach’s two-market illustration. Let x^i be demand for good i . Consider two goods. If we change p^1 first and then p^2 , the surplus change is

$$\Delta S_1 = - \int_{p_0^1}^{p_1^1} x^1(p^1, p_0^2) dp^1 - \int_{p_0^2}^{p_1^2} x^2(p_1^1, p^2) dp^2.$$

If instead we change p^2 first and then p^1 :

$$\Delta S_2 = - \int_{p_0^1}^{p_1^1} x^1(p^1, p_1^2) dp^1 - \int_{p_0^2}^{p_1^2} x^2(p_0^1, p^2) dp^2.$$

The difference is

$$\Delta S_2 - \Delta S_1 = - \int_{p_0^1}^{p_1^1} [x^1(p^1, p_1^2) - x^1(p^1, p_0^2)] dp^1 + \int_{p_0^2}^{p_1^2} [x^2(p_1^1, p^2) - x^2(p_0^1, p^2)] dp^2.$$

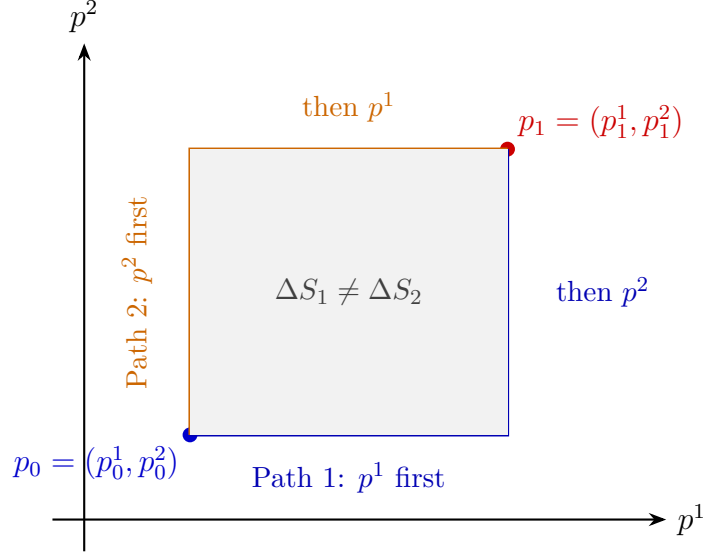


Figure 8: Path-dependence of Marshallian consumer surplus. The line integral $\Delta S = -\int \sum x^i dp^i$ from p_0 to p_1 depends on whether we change p^1 first (Path 1) or p^2 first (Path 2). The two values differ whenever $\partial x^1/\partial p^2 \neq \partial x^2/\partial p^1$.

For infinitesimal changes ($p_1^1 = p_0^1 + dp^1$, $p_1^2 = p_0^2 + dp^2$), this reduces to

$$\Delta S_2 - \Delta S_1 = \left(\frac{\partial x^2}{\partial p^1} - \frac{\partial x^1}{\partial p^2} \right) dp^1 dp^2,$$

which vanishes if and only if $\partial x^1/\partial p^2 = \partial x^2/\partial p^1$ —the exactness condition for the 1-form $\omega = -\sum_i x^i dp^i$.

B.2 Subsequent Literature

Several important developments followed Auerbach's survey, addressing the question of when and how well Marshallian surplus approximates the true welfare change.

Willig (1976). Showed that for a single price change, the percentage error from using Marshallian surplus in place of CV or EV is bounded by a simple function of the income elasticity η and the surplus-to-income ratio $\Delta S/I$. Specifically:

$$\frac{|\text{CV} - \Delta S|}{|\Delta S|} \leq \frac{|\eta|}{2} \frac{|\Delta S|}{I}.$$

For goods with small budget shares and moderate income elasticities, this error is negligible. This result was highly influential in applied work, as it justified using Marshallian surplus for single-market welfare calculations.

Hausman (1981). Rather than approximating, Hausman showed how to recover the exact Hicksian demand (and hence exact CV) from an estimated Marshallian demand system. The

method integrates the expenditure function using Roy's identity and the Slutsky equation. Hausman applied this to U.S. labour supply to show that the deadweight loss from income taxation was substantially larger than the Harberger triangle approximation suggested. The "exact consumer surplus" approach became standard in applied industrial organisation and public economics.

Chipman and Moore (1976, 1980). Studied the multi-price-change problem directly and proved a sharp impossibility result:

Proposition 17 (Chipman–Moore). *Consumer surplus provides a welfare ordering consistent with utility across all possible multi-price changes if and only if preferences admit a quasi-linear representation (zero income effects for all priced goods).*

This means there is no way to rescue consumer surplus as a general multi-price welfare measure without giving up income effects entirely. The result underscores that CV and EV are the only theoretically sound measures for general welfare evaluation.

McKenzie and Pearce (1976). Independently established that Marshallian surplus is path-independent if and only if the Marshallian cross-price Jacobian $\partial x^i / \partial p^j$ is symmetric—which, by the Slutsky equation, requires zero income effects (or the degenerate case of a single good).

Vives (1987). Extended the analysis to show that consumer surplus remains a valid local approximation for small price changes, even with multiple prices. The path-dependence error is second-order in the size of the price change: the Jacobian asymmetry $(\partial x^i / \partial p^j - \partial x^j / \partial p^i)$ is multiplied by the product $dp^i dp^j$. Thus, for small simultaneous price changes, the three measures CV, AV, and EV all agree to first order, and the path of integration becomes irrelevant to that order of approximation.

B.3 Implications for Applied Welfare Analysis

Scenario	Recommended measure	Justification
Single price change, small budget share	AV (Marshallian surplus)	Willig bounds: error is negligible
Single price change, exact welfare	CV or EV via Hausman method	Exact; no approximation error
Multiple price changes	CV or EV only	AV is path-dependent (Chipman–Moore)
Multiple small price changes	AV acceptable as approximation	Path-dependence is second-order (Vives)
Quasi-linear preferences	$AV = CV = EV$	All three coincide; no path-dependence

The overarching conclusion: modern applied welfare economics predominantly uses CV and EV rather than Marshallian surplus, especially when multiple prices change or when precision matters. The Marshallian surplus remains a useful shortcut for single-market, small-change analysis, but its theoretical foundations are limited.

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