

Summary of Willig (1976), “Consumer’s Surplus Without Apology”

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1 Overview and Contribution

Willig's paper settles a long-standing controversy about the validity of Marshallian consumer surplus as a welfare measure. The main result is a set of tight, computable bounds on the percentage error of approximating the compensating variation (CV) and equivalent variation (EV) with the observable Marshallian consumer surplus (AV). The bounds depend only on two observable quantities: the income elasticity of demand η and the surplus-to-income ratio $|AV|/I$.

In most practical applications, both quantities are small, so the approximation error is negligible—often smaller than the statistical error in estimating the demand curve itself. This justifies the “unapologetic” use of consumer surplus in applied welfare economics.

2 Setup and Notation

2.1 Notation Map

The summary uses the lecture notes (EC 121a) notation throughout. For reference, the correspondence to Willig's original notation is:

Willig	Lecture Notes	Description
$l(p, m)$	$v(p, I)$	Indirect utility function
$\mu(p \mid p^0, m^0)$	$e(p, v(p^0, I))$	Income compensation (expenditure) function
C	CV	Compensating variation
E	EV	Equivalent variation
A	AV	Marshallian consumer surplus (area variation)
m^0	I	Consumer's base income
$X^i(p, m)$	$x_i(p, I)$	Marshallian demand for good i
η	η	Income elasticity of demand

2.2 Consumer's Problem

The consumer maximises an increasing, strictly quasi-concave, ordinal utility function $U(x)$ subject to the budget constraint $\sum_i p_i x_i = I$. The resulting Marshallian demand functions are $x_i(p, I)$, assumed differentiable.

Definition 2.1 (Indirect utility function).

$$v(p, I) \equiv U[x_1(p, I), x_2(p, I), \dots, x_n(p, I)].$$

By nonsatiation, v is strictly increasing in I and decreasing in each p_i .

3 Compensating and Equivalent Variations

Let the initial situation be (p^0, I) and the alternative be (p', I') .

Definition 3.1 (Compensating variation). *The compensating variation CV is the income adjustment that makes the consumer indifferent to the price change:*

$$v(p^0, I) = v(p', I + CV). \quad (1)$$

The welfare test is:

$$v(p', I') \gtrless v(p^0, I) \quad \text{as} \quad I' - I \gtrless \text{CV}.$$

Definition 3.2 (Equivalent variation). *The equivalent variation EV is defined by:*

$$v(p^0, I - EV) = v(p', I). \quad (2)$$

In words, $-EV$ is the income change at the old prices that has the same welfare impact as the price change from p^0 to p' .

4 The Expenditure Function and Consumer's Surplus

Definition 4.1 (Expenditure function). *The expenditure function $e(p, \bar{u})$ gives the minimum income the consumer needs at prices p to achieve utility level $\bar{u}$. Evaluated at $\bar{u} = v(p^0, I)$:*

$$v[p, e(p, v(p^0, I))] = v(p^0, I). \quad (3)$$

Trivially, $e(p^0, v(p^0, I)) = I$. The CV and EV can be expressed as:

$$\text{CV} = e(p', v(p^0, I)) - e(p^0, v(p^0, I)) = e(p', v(p^0, I)) - I, \quad (4)$$

$$\text{EV} = e(p', v(p', I)) - e(p^0, v(p', I)) = I - e(p^0, v(p', I)). \quad (5)$$

4.1 Shephard's Lemma

By the envelope theorem (Shephard's lemma):

$$\frac{\partial e(p, v(p^0, I))}{\partial p_i} = x_i(p, e(p, v(p^0, I))). \quad (6)$$

This is the Hicksian (compensated) demand at the utility level $v(p^0, I)$.

4.2 Integral Representations of CV, EV, and AV

Restricting attention to a change in a single price p_1 , with $p^0 = (p_1^0, p_2^0, \dots, p_n^0)$ and $p' = (p_1', p_2^0, \dots, p_n^0)$, the Fundamental Theorem of Calculus and (6) give:

$$\text{CV} = \int_{p_1^0}^{p_1'} x_1(p_1, p_{-1}^0, e((p_1, p_{-1}^0), v(p^0, I))) dp_1, \quad (7)$$

$$\text{EV} = \int_{p_1^0}^{p_1'} x_1(p_1, p_{-1}^0, e((p_1, p_{-1}^0), v(p', I))) dp_1, \quad (8)$$

$$\text{AV} \equiv \int_{p_1^0}^{p_1'} x_1(p_1, p_{-1}^0, I) dp_1. \quad (9)$$

Here (7) and (8) are areas under *Hicksian* demand curves (compensated to $v(p^0, I)$ and $v(p', I)$ respectively), while (9) is the area under the *Marshallian* demand curve. The sign convention is: $\text{AV} > 0$ for a price increase ($p_1' > p_1^0$) and $\text{AV} < 0$ for a price decrease.

4.3 Qualitative Ordering

For a noninferior good ($\partial x_1 / \partial I \geq 0$) and a price increase ($p'_1 > p_1^0$):

$$CV \geq AV \geq EV.$$

The inequalities reverse for an inferior good. If there is no income effect ($\partial x_1 / \partial I \equiv 0$), then $CV = AV = EV$. (This corresponds to Proposition 16 of the lecture notes.)

5 Section III: Constant Income Elasticity — Exact Computation

This section is the analytical core that motivates the general bounds.

Notational shorthand. To keep the ODE derivations readable, we write $\tilde{e}(p_1)$ for the expenditure function $e((p_1, p_{-1}^0), v(p^0, I))$ viewed as a function of p_1 alone (with p_{-1}^0 and the reference utility $v(p^0, I)$ held fixed).

5.1 Assumption

The income elasticity of demand for good 1 is constant:

$$\eta \equiv \frac{\partial x_1(p, I)}{\partial I} \frac{I}{x_1(p, I)}, \quad (10)$$

which is independent of I (though it may depend on p).

Lemma 5.1 (Demand under constant income elasticity). *If (10) holds, then*

$$x_1(p, I') = x_1(p, I) \left(\frac{I'}{I} \right)^\eta \quad (11)$$

for any income level I' .

Proof. Fix prices p and treat demand as a function of income alone. The definition of constant income elasticity (10) says:

$$\frac{\partial x_1(p, I')}{\partial I'} \frac{I'}{x_1(p, I')} = \eta \quad \text{for all } I' > 0.$$

Rearranging, this is the separable ODE:

$$\frac{1}{x_1} \frac{dx_1}{dI'} = \frac{\eta}{I'}, \quad \text{i.e.,} \quad \frac{dx_1}{x_1} = \eta \frac{dI'}{I'}.$$

Integrate both sides, with the left side running from $x_1(p, I)$ to $x_1(p, I')$ and the right side from I to I' :

$$\int_{x_1(p, I)}^{x_1(p, I')} \frac{dx_1}{x_1} = \eta \int_I^{I'} \frac{dI'}{I'}.$$

Evaluating:

$$\log x_1(p, I') - \log x_1(p, I) = \eta [\log I' - \log I],$$

i.e.,

$$\log \frac{x_1(p, I')}{x_1(p, I)} = \eta \log \frac{I'}{I}.$$

Exponentiating both sides:

$$\frac{x_1(p, I')}{x_1(p, I)} = \left(\frac{I'}{I} \right)^\eta,$$

which gives (11). □

Remark 5.2. *The result that $\frac{x_1(p, I')}{x_1(p, I)} = \left(\frac{I'}{I} \right)^\eta$ can be interpreted naturally as the left hand side is percentage increase of demand and (I'/I) is the percentage increase of income.*

5.2 Derivation of the Expenditure Function

Substituting (11) into Shephard's lemma (6), using the shorthand $\tilde{e} = \tilde{e}(p_1)$:

$$\frac{d\tilde{e}}{dp_1} = x_1(p_1, \tilde{e}) = x_1(p_1, I) \left(\frac{\tilde{e}}{I} \right)^\eta. \quad (12)$$

This is a separable ODE. Rearranging:

$$\tilde{e}^{-\eta} d\tilde{e} = I^{-\eta} x_1(p_1, I) dp_1.$$

Case $\eta \neq 1$. Integrating both sides from p_1^0 to p_1' , with initial condition $\tilde{e}(p_1^0) = I$:

$$\text{Left side: } \int_I^{\tilde{e}(p_1')} \tilde{e}^{-\eta} d\tilde{e} = \frac{[\tilde{e}(p_1')]^{1-\eta} - I^{1-\eta}}{1-\eta}. \quad (13)$$

$$\text{Right side: } I^{-\eta} \int_{p_1^0}^{p_1'} x_1(p_1, I) dp_1 = I^{-\eta} \text{AV}. \quad (14)$$

Equating (13) and (14) and solving for $\tilde{e}(p_1')$:

$$\boxed{e(p', v(p^0, I)) = I \left[1 + (1-\eta) \frac{\text{AV}}{I} \right]^{1/(1-\eta)}}. \quad (15)$$

This is Willig's equation (15).

Case $\eta = 1$. When $\eta = 1$, the separable ODE (12) becomes:

$$\frac{d\tilde{e}}{dp_1} = x_1(p_1, I) \frac{\tilde{e}}{I}.$$

Dividing both sides by $\tilde{e}$ (which is positive):

$$\frac{1}{\tilde{e}} \frac{d\tilde{e}}{dp_1} = \frac{x_1(p_1, I)}{I}.$$

The left side is a total derivative: $\frac{1}{\tilde{e}} \frac{d\tilde{e}}{dp_1} = \frac{d}{dp_1} [\log \tilde{e}]$. So the equation reads:

$$\frac{d}{dp_1} [\log \tilde{e}] = \frac{x_1(p_1, I)}{I}.$$

Integrate both sides from p_1^0 to p_1' :

$$\int_{p_1^0}^{p_1'} \frac{d}{dp_1} [\log \tilde{e}] dp_1 = \frac{1}{I} \int_{p_1^0}^{p_1'} x_1(p_1, I) dp_1.$$

The left side, by the fundamental theorem of calculus with $\tilde{e}(p_1^0) = I$, equals:

$$\log \tilde{e}(p_1') - \log I = \log \frac{\tilde{e}(p_1')}{I}.$$

The right side equals AV/I . Hence:

$$\log \frac{\tilde{e}(p_1')}{I} = \frac{AV}{I}.$$

Exponentiating both sides:

$$\frac{\tilde{e}(p_1')}{I} = \exp \left[\frac{AV}{I} \right],$$

and multiplying through by I :

$$e(p', v(p^0, I)) = I \exp \left[\frac{AV}{I} \right]. \quad (16)$$

This is Willig's equation (16).

Remark 5.3. Equation (16) is also the limit of (15) as $\eta \rightarrow 1$. To see this, write $\alpha = 1 - \eta$ and note that as $\alpha \rightarrow 0$:

$$\left[1 + \alpha \frac{AV}{I} \right]^{1/\alpha} \rightarrow \exp \left[\frac{AV}{I} \right],$$

since $\lim_{\alpha \rightarrow 0} (1 + \alpha t)^{1/\alpha} = e^t$ for any constant t .

5.3 Exact CV and EV

From (4) and (15) (for $\eta \neq 1$):

$$CV = I \left\{ \left[1 + (1 - \eta) \frac{AV}{I} \right]^{1/(1-\eta)} - 1 \right\}. \quad (17)$$

For the equivalent variation, one reverses the roles of p^0 and p' in the integration, which replaces AV by $-AV$ and changes the reference utility. The analogous formula is:

$$EV = I \left\{ 1 - \left[1 - (1 - \eta) \frac{AV}{I} \right]^{1/(1-\eta)} \right\}. \quad (18)$$

5.4 Taylor Approximation

The key Taylor expansion is: for small t ,

$$(1 + t)^{1/(1-\eta)} \approx 1 + \frac{t}{1-\eta} + \frac{\eta t^2}{2(1-\eta)^2}.$$

Applying this with $t = (1 - \eta) AV/I$ (so that $t/(1 - \eta) = AV/I$):

$$CV \approx AV + \frac{\eta AV^2}{2I}, \quad (19)$$

$$EV \approx AV - \frac{\eta AV^2}{2I}. \quad (20)$$

Hence the percentage errors are:

$$\frac{CV - AV}{|AV|} \approx \frac{\eta AV}{2I}, \quad \frac{AV - EV}{|AV|} \approx \frac{\eta AV}{2I}. \quad (21)$$

6 Section IV: General Income Elasticity — The Main Theorem

6.1 Assumption

In the relevant region of price-income space, the income elasticity of demand for good 1 is bounded:

$$\underline{\eta} \leq \frac{\partial x_1(p, I')}{\partial I'} \frac{I'}{x_1(p, I')} \leq \bar{\eta}, \quad (22)$$

with $\underline{\eta}, \bar{\eta} \neq 1$.

6.2 Key Lemma: Demand Ratio Bounds

Lemma 6.1 (Demand ratio bounds, Willig's equation (18)). *Under the elasticity bounds (22), for all $I_2 \geq I_1 > 0$:*

$$\left(\frac{I_2}{I_1} \right)^{\underline{\eta}} \leq \frac{x_1(p, I_2)}{x_1(p, I_1)} \leq \left(\frac{I_2}{I_1} \right)^{\bar{\eta}}. \quad (23)$$

Proof. Define $f(t) = \log x_1(p, e^t)$. By the chain rule:

$$f'(t) = \frac{\partial x_1}{\partial I'} \Big|_{I'=e^t} \cdot \frac{e^t}{x_1(p, e^t)} = \frac{\partial x_1}{\partial I'} \frac{I'}{x_1(p, I')} = \eta(p, I').$$

The chain rule applies as follows. Write $I' = e^t$ and $g(I') = x_1(p, I')$, so $f(t) = \log g(e^t)$. The outer derivative gives $1/g(e^t)$; the inner derivative gives $g'(e^t) \cdot e^t = (\partial x_1 / \partial I') \cdot e^t$. Combining: $f'(t) = (\partial x_1 / \partial I') \cdot e^t / x_1(p, e^t) = \eta(p, I')$.

Let $t_1 = \log I_1$ and $t_2 = \log I_2$, so $t_2 \geq t_1$. By the fundamental theorem of calculus and the bounds $\underline{\eta} \leq f'(t) \leq \bar{\eta}$:

$$\underline{\eta} (t_2 - t_1) \leq \int_{t_1}^{t_2} f'(t) dt = f(t_2) - f(t_1) \leq \bar{\eta} (t_2 - t_1).$$

Substituting back:

$$\underline{\eta} \log \frac{I_2}{I_1} \leq \log \frac{x_1(p, I_2)}{x_1(p, I_1)} \leq \bar{\eta} \log \frac{I_2}{I_1}.$$

Since $\log(I_2/I_1) \geq 0$, exponentiating preserves the inequalities, yielding (23). $\square$

6.3 Differential Inequality Argument

The proof of the main bounds uses a comparison principle for the ODE (12).

Step 1: Sandwich the ODE. For a price increase ($p'_1 > p_1^0$), we have $\tilde{e}(p_1) \geq \tilde{e}(p_1^0) = I$ for $p_1 \geq p_1^0$. Setting $I_2 = \tilde{e}(p_1)$ and $I_1 = I$ in Lemma 6.1, and using Shephard's lemma $d\tilde{e}/dp_1 = x_1(p_1, \tilde{e})$:

$$x_1(p_1, I) \left(\frac{\tilde{e}}{I} \right)^\eta \leq \frac{d\tilde{e}}{dp_1} \leq x_1(p_1, I) \left(\frac{\tilde{e}}{I} \right)^{\bar{\eta}}. \quad (24)$$

Step 2: Separate variables (lower bound). From the left inequality of (24), divide both sides by $\tilde{e}^\eta$ (positive):

$$x_1(p_1, I) I^{-\eta} \leq \tilde{e}^{-\eta} \frac{d\tilde{e}}{dp_1}.$$

The factor $I^{-\eta}$ arises because $(\tilde{e}/I)^\eta / \tilde{e}^\eta = I^{-\eta}$.

The right side is a total derivative: for $\eta \neq 1$,

$$\tilde{e}^{-\eta} \frac{d\tilde{e}}{dp_1} = \frac{d}{dp_1} \left[\frac{\tilde{e}^{1-\eta}}{1-\eta} \right].$$

This follows because $\frac{d}{dp_1} \left[\frac{\tilde{e}^{1-\eta}}{1-\eta} \right] = \frac{(1-\eta)\tilde{e}^{-\eta}}{1-\eta} \cdot \frac{d\tilde{e}}{dp_1} = \tilde{e}^{-\eta} \frac{d\tilde{e}}{dp_1}$.

So the inequality becomes:

$$x_1(p_1, I) I^{-\eta} \leq \frac{d}{dp_1} \left[\frac{\tilde{e}^{1-\eta}}{1-\eta} \right]. \quad (25)$$

Step 3: Separate variables (upper bound). From the right inequality of (24), divide both sides by $\tilde{e}^{\bar{\eta}}$:

$$\frac{d}{dp_1} \left[\frac{\tilde{e}^{1-\bar{\eta}}}{1-\bar{\eta}} \right] \leq x_1(p_1, I) I^{-\bar{\eta}}. \quad (26)$$

Step 4: Integrate both inequalities. Integrating (25) over $[p_1^0, p'_1]$:

$$I^{-\eta} \int_{p_1^0}^{p'_1} x_1(p_1, I) dp_1 \leq \int_{p_1^0}^{p'_1} \frac{d}{dp_1} \left[\frac{\tilde{e}^{1-\eta}}{1-\eta} \right] dp_1.$$

The left side equals $I^{-\eta} \cdot \text{AV}$. The right side, by the fundamental theorem of calculus with $\tilde{e}(p_1^0) = I$, equals:

$$\frac{[\tilde{e}(p'_1)]^{1-\eta} - I^{1-\eta}}{1-\eta}.$$

Hence:

$$I^{-\eta} \cdot \text{AV} \leq \frac{[\tilde{e}(p'_1)]^{1-\eta} - I^{1-\eta}}{1-\eta}. \quad (27)$$

Similarly, integrating (26):

$$\frac{[\tilde{e}(p'_1)]^{1-\bar{\eta}} - I^{1-\bar{\eta}}}{1-\bar{\eta}} \leq I^{-\bar{\eta}} \cdot \text{AV}. \quad (28)$$

Step 5: Solve for $\tilde{e}(p'_1)$. We demonstrate the algebraic steps for the lower bound from (27); the upper bound from (28) is analogous.

Multiply (27) by $(1 - \underline{\eta})$:

$$(1 - \underline{\eta}) I^{-\underline{\eta}} \cdot AV \leq [\tilde{e}(p'_1)]^{1-\underline{\eta}} - I^{1-\underline{\eta}}.$$

(For $\underline{\eta} < 1$, the factor $1 - \underline{\eta} > 0$ and the inequality direction is preserved. For $\underline{\eta} > 1$, the factor is negative and the direction flips, but the final formula is the same.)

Add $I^{1-\underline{\eta}}$ to both sides:

$$I^{1-\underline{\eta}} + (1 - \underline{\eta}) I^{-\underline{\eta}} \cdot AV \leq [\tilde{e}(p'_1)]^{1-\underline{\eta}}.$$

Factor $I^{-\underline{\eta}}$ on the left, using $I^{1-\underline{\eta}} = I^{-\underline{\eta}} \cdot I$:

$$I^{-\underline{\eta}} [I + (1 - \underline{\eta}) AV] \leq [\tilde{e}(p'_1)]^{1-\underline{\eta}}.$$

Factor out I inside the bracket:

$$I^{1-\underline{\eta}} \left[1 + (1 - \underline{\eta}) \frac{AV}{I} \right] \leq [\tilde{e}(p'_1)]^{1-\underline{\eta}}.$$

Raise both sides to the power $1/(1 - \underline{\eta})$:

$$I \left[1 + (1 - \underline{\eta}) \frac{AV}{I} \right]^{1/(1-\underline{\eta})} \leq \tilde{e}(p'_1).$$

The upper bound from (28) follows identically, with $\bar{\eta}$ replacing $\underline{\eta}$, yielding $\tilde{e}(p'_1) \leq (\dots)$.

6.4 The Main Bounds

Theorem 6.2 (Willig's equation (19): Bounds on the expenditure function). *Under the income elasticity bounds (22) with $\underline{\eta}, \bar{\eta} \neq 1$, provided $1 + (1 - \underline{\eta}) AV/I > 0$ and $1 + (1 - \bar{\eta}) AV/I > 0$:*

$$\boxed{I \left[1 + (1 - \underline{\eta}) \frac{AV}{I} \right]^{1/(1-\underline{\eta})} \leq e(p', v(p^0, I)) \leq I \left[1 + (1 - \bar{\eta}) \frac{AV}{I} \right]^{1/(1-\bar{\eta})}}. \quad (29)$$

Remark 6.3. When $\underline{\eta} = \bar{\eta} = \eta$ (constant income elasticity), both bounds coincide and we recover the exact formula (15).

Remark 6.4 (Price decrease). For a price decrease from p_1^0 to p'_1 with $p'_1 < p_1^0$, we have $\tilde{e}(p_1) \leq I$ and $AV < 0$. The derivation proceeds by setting $I_2 = I$ and $I_1 = \tilde{e}(p_1)$ in Lemma 6.1 and following the same sequence of steps. The result is the same formula (29), now with $AV < 0$.

6.5 Bounds on the Percentage Error for CV

Using $CV = e(p', v(p^0, I)) - I$ and dividing by $|AV|$, equation (29) yields:

Corollary 6.5 (Willig's equation (20): Error bounds for CV).

$$\frac{\left[1 + (1 - \underline{\eta}) \frac{AV}{I} \right]^{1/(1-\underline{\eta})} - 1 - \frac{AV}{I}}{|AV|/I} \leq \frac{CV - AV}{|AV|} \leq \frac{\left[1 + (1 - \bar{\eta}) \frac{AV}{I} \right]^{1/(1-\bar{\eta})} - 1 - \frac{AV}{I}}{|AV|/I}. \quad (30)$$

Proof. From the lower bound of (29):

$$CV \geq I \left[1 + (1 - \underline{\eta}) \frac{AV}{I} \right]^{1/(1-\underline{\eta})} - I.$$

So:

$$CV - AV \geq I \left\{ \left[1 + (1 - \underline{\eta}) \frac{AV}{I} \right]^{1/(1-\underline{\eta})} - 1 \right\} - AV.$$

Dividing by $|AV|$ and noting $AV/|AV| = AV/I \cdot I/|AV|$:

$$\frac{CV - AV}{|AV|} \geq \frac{\left[1 + (1 - \underline{\eta}) \frac{AV}{I} \right]^{1/(1-\underline{\eta})} - 1 - \frac{AV}{I}}{|AV|/I}.$$

The upper bound follows identically from the upper bound of (29). $\square$

6.6 Bounds on the Percentage Error for EV

Corollary 6.6 (Willig's equation (21): Error bounds for EV). *By reversing the roles of p' and p^0 in (29) (but not in the definition of AV):*

$$\frac{\left[1 - (1 - \underline{\eta}) \frac{AV}{I} \right]^{1/(1-\underline{\eta})} - 1 + \frac{AV}{I}}{|AV|/I} \leq \frac{AV - EV}{|AV|} \leq \frac{\left[1 - (1 - \bar{\eta}) \frac{AV}{I} \right]^{1/(1-\bar{\eta})} - 1 + \frac{AV}{I}}{|AV|/I}. \quad (31)$$

Proof. The equivalent variation satisfies $EV = I - e(p^0, v(p', I))$. To bound $e(p^0, v(p', I))$, apply Theorem 6.2 with the roles of p^0 and p' swapped. The “area” in this reversed problem is $\int_{p_1^0}^{p_1^1} x_1(p_1, I) dp_1 = -AV$. Substituting $-AV$ for AV in (29) gives bounds on $e(p^0, v(p', I))$, and then $EV = I - e(p^0, v(p', I))$ yields (31). $\square$

7 The Rules of Thumb

The exact bounds (30) and (31) simplify dramatically when $|\eta AV/2I|$ is small.

Theorem 7.1 (Rules of thumb, Willig's equations (1) and (2)). *If $|\bar{\eta} AV/2I| < 0.05$, $|\underline{\eta} AV/2I| < 0.05$, and $|AV/I| < 0.9$, then:*

$$\frac{\underline{\eta} |AV|}{2I} \leq \frac{CV - AV}{|AV|} \leq \frac{\bar{\eta} |AV|}{2I}, \quad (32)$$

$$\frac{\underline{\eta} |AV|}{2I} \leq \frac{AV - EV}{|AV|} \leq \frac{\bar{\eta} |AV|}{2I}. \quad (33)$$

Derivation of (32) from Corollary 6.5

Write $a = AV/I$ for short. The lower bound in (30) has numerator

$$\left[1 + (1 - \underline{\eta}) a \right]^{1/(1-\underline{\eta})} - 1 - a.$$

Apply the Taylor expansion from Section 5 with $t = (1 - \underline{\eta})a$:

$$(1 + t)^{1/(1-\underline{\eta})} \approx 1 + \frac{t}{1-\underline{\eta}} + \frac{\underline{\eta} t^2}{2(1-\underline{\eta})^2}.$$

Since $t/(1-\underline{\eta}) = a$ and $t^2/(1-\underline{\eta})^2 = a^2$, the power expression becomes approximately $1 + a + \underline{\eta} a^2/2$. The terms $1 + a$ are not an approximation—they already appear in the exact formula with a minus sign. Subtracting:

$$(1 + a + \frac{\underline{\eta} a^2}{2}) - 1 - a = \frac{\underline{\eta} a^2}{2}.$$

This is why the error is second-order: the power expression and $1 + a$ agree to first order, so their difference is the quadratic residual $\underline{\eta} a^2/2$.

Dividing by $|a|$ (the denominator $|AV|/I$ in (30)):

$$\frac{\underline{\eta} a^2/2}{|a|} = \frac{\underline{\eta} |a|}{2} = \frac{\underline{\eta} |AV|}{2I}.$$

The identical calculation with $\bar{\eta}$ in the upper bound of (30) gives $\bar{\eta} |AV|/(2I)$.

Derivation of (33) from Corollary 6.6

The lower bound in (31) has numerator $[1 - (1 - \underline{\eta})a]^{1/(1-\underline{\eta})} - 1 + a$. This is the same calculation with a replaced by $-a$: set $t = -(1 - \underline{\eta})a$, so $t/(1 - \underline{\eta}) = -a$ and $t^2/(1 - \underline{\eta})^2 = a^2$. The Taylor expansion gives approximately $1 - a + \underline{\eta} a^2/2$. Subtracting $1 - a$ (i.e., the “ $-1 + a$ ” in the numerator):

$$(1 - a + \frac{\underline{\eta} a^2}{2}) - 1 + a = \frac{\underline{\eta} a^2}{2}.$$

Since a enters squared, the sign change does not matter, and we obtain the same simplified bound $\underline{\eta} |AV|/(2I)$.

Role of the numerical conditions

The derivation above uses a second-order Taylor approximation, so there is a remainder term. The conditions $|\bar{\eta} AV/2I| < 0.05$, $|\underline{\eta} AV/2I| < 0.05$, and $|AV/I| < 0.9$ are not derived analytically. Instead, Willig verifies numerically in his Table 1 that under these conditions, the exact bounds (30)–(31) and the simple expressions $\eta |a|/2$ differ by less than 0.005.

Corollary 7.2 (Proximity of CV and EV, Willig’s equation (23)). *Adding (32) and (33):*

$$\frac{\underline{\eta} |AV|}{I} \leq \frac{CV - EV}{|AV|} \leq \frac{\bar{\eta} |AV|}{I}. \quad (34)$$

8 Numerical Example

Suppose the measured income elasticity of demand is $\eta = 0.8$ (so $\underline{\eta} = \bar{\eta} = 0.8$), and the consumer surplus area between the old and new prices is $AV = 0.05 I$ (5% of income).

Then:

$$\frac{\eta |AV|}{2I} = \frac{0.8 \times 0.05}{2} = 0.02.$$

So the compensating variation is within 2% of the measured consumer surplus, and similarly for the equivalent variation. This percentage error is likely smaller than the statistical uncertainty in the demand curve estimate itself.

9 Section V: Individual Welfare and Consumer's Surplus

Willig concludes by applying the bounds to welfare tests. The question is: can we use AV —which is computable from Marshallian demand data—to make correct welfare judgments, even though the theoretically correct object is CV ?

By definition of CV , $v(p', I + CV) = v(p^0, I)$. Since v is strictly increasing in income, the welfare comparison reduces to:

$$v(p', I') > v(p^0, I) \iff I' - I > CV.$$

The problem is that CV is unobservable (it requires the Hicksian demand curve). However, from Corollary 6.5 or the rules of thumb, CV is bracketed by computable bounds $\underline{CV} \leq CV \leq \overline{CV}$. This yields two safe conclusions:

$$\begin{aligned} I' - I > \overline{CV} &\implies I' - I > CV \implies v(p', I') > v(p^0, I), \\ I' - I < \underline{CV} &\implies I' - I < CV \implies v(p', I') < v(p^0, I). \end{aligned}$$

If $\underline{CV} \leq I' - I \leq \overline{CV}$, the test is inconclusive. But when $|\eta AV/2I|$ is small, both bounds are approximately equal to AV , so the inconclusive gap is negligible and the test simplifies to:

$$v(p', I') \gtrless v(p^0, I) \quad \text{as} \quad I' - I \gtrless AV.$$

This justifies cost-benefit welfare analysis at the individual level using the observable Marshallian consumer surplus.

9.1 When Does the Marshallian Approximation Work Well?

Recall the Willig (1976) bound on the percentage error from using the area variation AV in place of the compensating variation CV :

$$\frac{|CV - AV|}{|AV|} \approx \frac{\eta |AV|}{2I},$$

where η is the income elasticity of demand and I is income. The error is small when (i) the income elasticity η is small, and (ii) the welfare change AV is small relative to income. For most individual consumer goods, both conditions hold comfortably, so the Marshallian consumer surplus is an excellent approximation to the theoretically correct Hicksian measures. The following examples illustrate when the approximation does and does not work well.

Small Error: Low η and Low AV/I

- **Salt.** Estimated income elasticity is near zero ($\eta \approx 0.05$ – 0.1). Annual consumer surplus from salt is at most a few dollars, so $|AV|/I$ is negligible. Even for a large price change, the Willig error is on the order of 0.001%.
- **Toothpaste, matches, basic condiments.** These are inexpensive necessities with very low income elasticities. Both η and $|AV|/I$ are tiny, so CV , Δ , and AV are virtually identical.
- **Urban bus fare.** Income elasticity is typically estimated at $\eta \approx 0.2$ – 0.5 . A fare change might generate $|AV| \approx \$30$ – $\$80$ per year for a typical rider. Against an income of $I = \$60,000$, the Willig error is well under 0.1%.

Large Error: High η and/or High AV/I

- **Housing (rent).** Income elasticity is moderate ($\eta \approx 0.7$ – 1.0), but the welfare change can be large relative to income. A rent increase generating $|AV| \approx \$5,000$ against $I = \$60,000$ gives $|AV|/I \approx 0.08$. The Willig error can reach 3–4%, which is why applied welfare analyses of housing policy sometimes use Hausman-style exact CV calculations.
- **Luxury automobiles.** Income elasticity is high ($\eta \approx 2$ – 3). The surplus change from a price shift on a vehicle costing \$50,000+ is non-trivial relative to income. The approximation error can easily reach 5–10%.
- **International leisure travel.** Income elasticity estimates range from 1.5 to 3.0. A family devoting a significant share of income to vacations may have $|AV|/I \approx 0.03$ – 0.05 , pushing the error to several percent.
- **Private schooling or higher education.** High income elasticity combined with expenditure that can be 10–20% of household income makes this a case where the wedge between CV and AV is economically meaningful.

Takeaway

For any single ordinary consumer good—the kind that occupies a small share of the budget—the Willig error is almost always negligible. The approximation breaks down for goods that are simultaneously income-elastic *and* account for a large budget share: big-ticket items or broad expenditure categories like “housing” or “food as a whole.” This is partly why Hausman (1981) chose to work with a broad expenditure category (all food, via the demand for rice) rather than a narrow commodity—the welfare stakes of getting CV right are higher when the budget share is large.

10 Summary: Why the Paper Matters

The lecture notes (Proposition 16) establish the qualitative ordering $CV \geq AV \geq EV$ for normal goods but provide no quantitative control over the gaps. Willig’s contribution is to show:

- (i) The percentage gap between AV and either CV or EV is approximately $\eta |AV|/(2I)$.
- (ii) Both η (income elasticity) and $|AV|/I$ (surplus-to-income ratio) are directly observable from Marshallian demand data.
- (iii) Since income elasticities typically cluster near 1 and consumer surplus areas are typically a small fraction of income, the product $\eta |AV|/(2I)$ is usually negligible.
- (iv) Even when the approximation is not tight, the exact bounds (29)–(31) allow computation of CV and EV from observables.

The paper thus bridges the gap between the theoretically correct but unobservable welfare measures (CV and EV) and the practically computable but theoretically suspect Marshallian surplus (AV).