

Handout: Hausman's (1981) Method for Exact Consumer's Surplus and Deadweight Loss

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1 Background and What Is New in Hausman (1981)

1.1 What Was Already Known

By 1981, the theoretical foundations for exact welfare measurement were well established:

- The **integrability theorem**—that one can recover the expenditure function from the Marshallian demand system by solving a system of PDEs, provided Slutsky symmetry holds—was established by Hurwicz and Uzawa (1971) and Samuelson (1950). The fundamental equation $\partial e / \partial p_i = x_i^*(p, e)$ for all $i = 1, \dots, N$ was known; what was required was the full system of N demand functions and verification of the Slutsky integrability conditions across all pairs of goods.
- The **duality theory** connecting the indirect utility function, the expenditure function, Roy's identity, and Shephard's lemma was standard (McKenzie, 1957; Diewert, 1974; Diamond and McFadden, 1974).
- **Willig (1976)** showed that the Marshallian consumer surplus (AV) is often a good approximation to the compensating variation (CV) and equivalent variation (EV), with percentage errors bounded by a simple function of the income elasticity and the surplus-to-income ratio.

Despite this theoretical knowledge, applied economists continued to use the Marshallian approximation, partly because the exact calculations were perceived as requiring estimation of the complete demand system. As Varian (1978, p. 210) wrote, the Hicksian demand curves are “unobservable” and the exact expressions “do not appear to be useful.”

1.2 Hausman's Contributions

Hausman (1981) made five distinct contributions:

1. Reducing the system of PDEs to a single ODE. The integrability literature required solving the full system of N PDEs $\partial e / \partial p_i = x_i^*(p_1, \dots, p_N, e)$ for $i = 1, \dots, N$, which requires all N demand functions and verification of Slutsky symmetry $s_{ij} = s_{ji}$ across all pairs. Hausman observed that if only *one* price changes, only a single ODE needs to be solved: $de / dp_1 = x_1^*(p_1, e)$. A single ODE always has a solution (by the standard existence theorem), so no Slutsky conditions need to be checked, and only one demand equation needs to be estimated.

2. Closed-form solutions for empirically common specifications. The prior literature established the *existence* of solutions; Hausman actually *solved* the ODE for the functional forms that applied economists routinely estimate:

- Linear demand (his eq. 12): $x_1 = \alpha p_1 + \delta y + z\gamma$
- Log-linear / constant elasticity demand (his eq. 20): $x_1 = e^{z\gamma} p_1^\alpha y^\delta$
- Quadratic-in-price demand (his eq. 30)

- Fully quadratic demand in both price and income (his eq. 32), solved via a transformation to the Schrödinger wave equation

For each specification, he provided explicit, ready-to-use formulas for the expenditure function, indirect utility function, and exact CV.

3. The “quasi” expenditure function and Hicks’ composite commodity theorem.

In the many-good case with a single price change, Hausman showed that one does not need the complete demand system. By applying Hicks’ aggregation theorem, all other goods collapse into a composite numeraire, reducing the N -good problem to the two-good case. The resulting “quasi” expenditure function gives the *exact same* CV as the true expenditure function for a single price change. This justified using a single estimated demand equation for exact welfare analysis.

4. The deadweight loss critique—the paper’s most influential contribution. Hausman demonstrated that even when Willig’s approximation works well for the *level* of consumer surplus (CV vs. AV differ by only 3.2% in his gasoline example), it can fail badly for *deadweight loss* (32% error). See Section 5 for the detailed geometric explanation using Hausman’s Figure 2.

5. Standard errors for welfare measures. Hausman pointed out that since the demand parameters are econometrically estimated, his closed-form expressions allow computation of standard errors for CV and DWL via the delta method. This was a practical innovation—prior welfare analysis reported point estimates without uncertainty quantification.

2 The Key Idea: Roy’s Identity as an ODE

2.1 Review of Approach used in integrability theorem: Shepard’s Lemma

Consider changing only the price of good 1, holding $p_{-1} = (p_2, \dots, p_N)$ fixed. By Shepard’s lemma and the duality relation $x_1^h(p_1, p_{-1}, \bar{u}) = x_1^*(p_1, p_{-1}, e(p_1, p_{-1}, \bar{u}))$, we obtain:

$$\boxed{\frac{\partial e(p_1, p_{-1}, \bar{u})}{\partial p_1} = x_1^*(p_1, p_{-1}, e(p_1, p_{-1}, \bar{u}))}. \quad (1)$$

This is an ODE in p_1 for the unknown function $e(\cdot)$, where the right-hand side involves only the *observed Marshallian demand*. The initial condition is $e(p_1^0, p_{-1}, \bar{u}) = I$, where $\bar{u} = v(p_1^0, p_{-1}, I)$.

Remark 2.1. No knowledge of x_1^h , e , or u is needed *a priori*. Everything is recovered by solving this single ODE. This is the same fundamental equation as in the Hurwicz–Uzawa integrability theory, but restricted to a single price direction. The key simplification is that a single ODE always has a solution, whereas the full system of N PDEs requires the integrability (Slutsky symmetry) condition.

2.2 Hausman's Route: The Indirect Utility ODE

Hausman's actual derivation takes a slightly different but equivalent route. He uses Roy's identity to derive an ODE for y (income) as a function of p_1 *along an indifference curve*.

Along a path of price change that keeps utility constant at u^0 , the total derivative of $v(p_1(t), y(t)) = u^0$ gives:

$$\frac{\partial v}{\partial p_1} \frac{dp_1}{dt} + \frac{\partial v}{\partial y} \frac{dy}{dt} = 0.$$

By Roy's identity, $x_1^* = -(\partial v / \partial p_1) / (\partial v / \partial y)$, so:

$$\frac{dy}{dp_1} = x_1^*(p_1, y). \quad (2)$$

This is the same ODE as (1)—the function $y(p_1)$ that keeps utility constant is precisely the expenditure function $e(p_1, \bar{u})$. Hausman solves this ODE to obtain the indirect utility function (by finding the constant of integration c and setting $c = u^0$), then inverts to get the expenditure function.

2.3 Recovering Exact CV and EV

Once the expenditure function is obtained, the exact welfare measures follow:

$$CV = e(p_1^1, p_{-1}, u^0) - I, \quad (3)$$

$$EV = I - e(p_1^0, p_{-1}, u^1). \quad (4)$$

3 Application 1: Linear Demand

3.1 Setup

Suppose the Marshallian demand for good 1 is linear in price and income (Hausman's eq. 12):

$$x_1^*(p_1, y) = \alpha p_1 + \delta y + z\gamma, \quad (5)$$

where $\alpha < 0$ (law of demand), $\delta > 0$ (normal good), and $z\gamma$ captures socioeconomic characteristics. Here p_1 and y are normalized by the price of the composite good p_2 .

3.2 Solving the ODE: The Integrating Factor Method

Substituting (5) into the ODE (2):

$$\frac{dy}{dp_1} = \alpha p_1 + \delta y + z\gamma. \quad (6)$$

This is a first-order linear ODE in y as a function of p_1 . We solve it step by step using the integrating factor method.

Step 1: Standard form. Move the δy term to the left-hand side:

$$\frac{dy}{dp_1} - \delta y = \alpha p_1 + z\gamma.$$

This has the form $y' + P(p_1)y = Q(p_1)$ with $P(p_1) = -\delta$ and $Q(p_1) = \alpha p_1 + z\gamma$.

Step 2: The integrating factor. For any first-order linear ODE in this form, multiplying by $\mu(p_1) = \exp(\int P(p_1) dp_1)$ converts the LHS into an exact derivative. Here:

$$\mu(p_1) = e^{\int (-\delta) dp_1} = e^{-\delta p_1}.$$

Multiply both sides of the standard-form ODE by $\mu(p_1)$:

$$e^{-\delta p_1} \frac{dy}{dp_1} - \delta e^{-\delta p_1} y = (\alpha p_1 + z\gamma) e^{-\delta p_1}.$$

Step 3: Recognize the LHS as an exact derivative. By the product rule:

$$\frac{d}{dp_1} [e^{-\delta p_1} y] = e^{-\delta p_1} \frac{dy}{dp_1} - \delta e^{-\delta p_1} y.$$

So the equation becomes:

$$\frac{d}{dp_1} [e^{-\delta p_1} y] = (\alpha p_1 + z\gamma) e^{-\delta p_1}.$$

Step 4: Integrate both sides. The LHS integrates trivially. For the RHS, use integration by parts on the $\alpha p_1 e^{-\delta p_1}$ term:

$$\begin{aligned} \int \alpha p_1 e^{-\delta p_1} dp_1 &= -\frac{\alpha}{\delta} p_1 e^{-\delta p_1} - \frac{\alpha}{\delta^2} e^{-\delta p_1}, \\ \int z\gamma e^{-\delta p_1} dp_1 &= -\frac{z\gamma}{\delta} e^{-\delta p_1}. \end{aligned}$$

Therefore:

$$e^{-\delta p_1} y = -\frac{\alpha}{\delta} p_1 e^{-\delta p_1} - \frac{\alpha}{\delta^2} e^{-\delta p_1} - \frac{z\gamma}{\delta} e^{-\delta p_1} + c.$$

Step 5: Solve for y . Multiply through by $e^{\delta p_1}$:

$$y(p_1) = c e^{\delta p_1} - \frac{1}{\delta} \left(\alpha p_1 + \frac{\alpha}{\delta} + z\gamma \right), \quad (7)$$

where c is the constant of integration, which depends on the indifference curve.

Hausman chooses $c = u^0$ as the cardinal utility index. Solving (7) for c gives the **indirect utility function** (Hausman's eq. 16):

$$\boxed{v(p_1, y) = e^{-\delta p_1} \left[y + \frac{1}{\delta} \left(\alpha p_1 + \frac{\alpha}{\delta} + z\gamma \right) \right]}. \quad (8)$$

Interchanging y and $\bar{u}$ gives the **expenditure function** (Hausman's eq. 17):

$$\boxed{e(p_1, \bar{u}) = e^{\delta p_1} \bar{u} - \frac{1}{\delta} \left(\alpha p_1 + \frac{\alpha}{\delta} + z\gamma \right)}. \quad (9)$$

Remark 3.1 (Why the constant of integration c is a utility level). The ODE $dy/dp_1 = x_1^*(p_1, y)$ was derived from $dv = 0$, so every solution curve $y(p_1)$ lies on a single indifference curve. The general solution

$$y(p_1) = c e^{\delta p_1} - \frac{1}{\delta} \left(\alpha p_1 + \frac{\alpha}{\delta} + z\gamma \right)$$

is a one-parameter family of such curves. By the ODE uniqueness theorem, distinct values of c correspond to distinct indifference curves, so c separates utility levels and is therefore a valid cardinal utility index.

Imposing the initial condition $y(p_1^0) = y^0$ pins down c uniquely:

$$c = e^{-\delta p_1^0} \left[y^0 + \frac{1}{\delta} \left(\alpha p_1^0 + \frac{\alpha}{\delta} + z\gamma \right) \right].$$

This expression is consistent with the expression of the indirect utility.

Hausman *defines* $u^0 := c$; this expression is then the indirect utility function $v(p_1^0, y^0)$. No pre-existing knowledge of v is assumed—the formula for v is a *output* of the ODE solution, not an input. The expenditure function follows immediately by reading the general solution as $e(p_1, \bar{u}) = \bar{u} e^{\delta p_1} - \frac{1}{\delta} (\alpha p_1 + \frac{\alpha}{\delta} + z\gamma)$, i.e., treating $c = \bar{u}$ as the argument and y as the value.

3.3 Verification: Slutsky Condition

The compensated demand follows from Shephard's lemma:

$$h_1(p_1, \bar{u}) = \frac{\partial e}{\partial p_1} = \delta e^{\delta p_1} \bar{u} - \frac{\alpha}{\delta}.$$

The Slutsky term is (Hausman's eq. 18):

$$s_{11} = \frac{\partial h_1}{\partial p_1} = \alpha + \delta(\alpha p_1 + \delta y + z\gamma) = \alpha + \delta x_1^* \leq 0, \quad (10)$$

which must be negative for the solution to be valid.

3.4 Exact CV for Linear Demand

The exact compensating variation (Hausman's eq. 19) is:

$$\boxed{\text{CV}(p_1^0, p_1^1, y^0) = \frac{1}{\delta} e^{\delta(p_1^1 - p_1^0)} \left[x_1^*(p_1^0, y^0) + \frac{\alpha}{\delta} \right] - \frac{1}{\delta} \left[x_1^*(p_1^1, y^0) + \frac{\alpha}{\delta} \right]}. \quad (11)$$

Remark 3.2 (Connection to the Marshallian measure). When the income effect is zero ($\delta = 0$), the exponential $e^{\delta(p_1^1 - p_1^0)} \rightarrow 1$ and the exact CV collapses to the Marshallian consumer surplus:

$$\text{AV} = \int_{p_1^0}^{p_1^1} (\alpha p_1 + z\gamma) dp_1 = \frac{\alpha}{2} (p_1^{12} - p_1^{02}) + z\gamma(p_1^1 - p_1^0).$$

This confirms the general result that $\text{CV} = \text{AV}$ when income effects vanish.

4 Application 2: Log-Linear (Constant Elasticity) Demand

4.1 Setup and Solution

Suppose the Marshallian demand is log-linear (Hausman's eq. 20):

$$x_1^*(p_1, y) = e^{z\gamma} p_1^\alpha y^\delta, \quad \delta \neq 1, \quad (12)$$

where $\alpha < 0$ is the (constant) price elasticity and $\delta > 0$ is the income elasticity.

The ODE (2) becomes separable: $y^{-\delta} dy = e^{z\gamma} p_1^\alpha dp_1$. Integrating with initial condition $y(p_1^0) = I$ and choosing the constant of integration as $c = u^0$, Hausman obtains the **indirect utility function** (his eq. 21):

$$v(p_1, y) = -e^{z\gamma} \frac{p_1^{1+\alpha}}{1+\alpha} + \frac{y^{1-\delta}}{1-\delta}, \quad (13)$$

and the **expenditure function** (his eq. 22):

$$e(p_1, \bar{u}) = \left[(1-\delta) \left(\bar{u} + e^{z\gamma} \frac{p_1^{1+\alpha}}{1+\alpha} \right) \right]^{1/(1-\delta)}. \quad (14)$$

Remark 4.1. For the special case $\delta = 1$ (unit income elasticity), the indirect utility function involves $\log y$ instead of $y^{1-\delta}/(1-\delta)$, and the expenditure function takes an exponential form (Hausman's footnote 18).

5 Exact Deadweight Loss: Hausman's Central Critique

5.1 Definition and Two Conventions for Tax Revenue

Consider a per-unit tax τ on good 1, raising the consumer price from p_1^0 to $p_1^1 = p_1^0 + \tau$. The deadweight loss is the welfare cost in excess of tax revenue collected. Two conventions for tax revenue appear in the literature, and they give two different but related DWL measures.

The tax revenue is computed using the actual (Marshallian) quantity demanded at the post-tax price:

$$R = \tau \cdot x_1^*(p_1^1, y^0) = \tau \cdot x^1.$$

Marshallian convention.

$$\text{DWL}_M = \text{AV} - R = \int_{p_1^0}^{p_1^1} [x_1(p, y^0)] dp - R. \quad (15)$$

Hicksian (exact) convention.

$$\text{DWL}_{\text{exact}} = \text{CV} - R = \int_{p_1^0}^{p_1^1} [h_1(p, u^0)] dp - R. \quad (16)$$

The Hicksian convention is standard in optimal taxation theory (Diamond–Mirrlees) because it isolates the pure efficiency cost of distorting relative prices, holding utility fixed at u^0 . It is the convention Hausman uses, and the one we adopt below.

5.2 Geometric Interpretation: Hausman's Figure 2

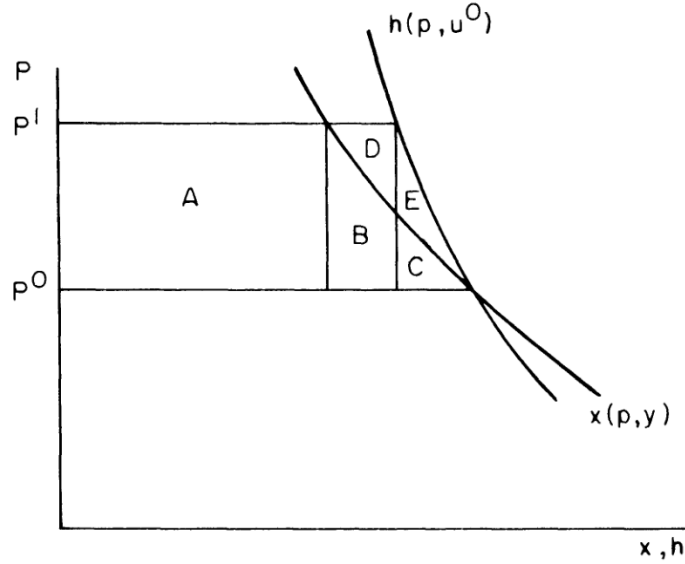


FIGURE 2

Figure 1: Hausman's Figure 2. The Hicksian compensated demand curve $h(p, u^0)$ (steeper, in (q, p) -space) and the Marshallian demand curve $x_1(p, y^0)$ (flatter) for a normal good. The price increases from p^0 to p^1 . Both curves pass through (x^0, p^0) . At p^1 , the Marshallian quantity x^1 is to the *left* of the Hicksian quantity h^1 (i.e., $x^1 < h^1$).

In Figure 1, the areas decompose as

$$\text{Marshallian AV} = \int_{p_1^0}^{p_1^1} x_1(p, y^0) dp = A + B + C, \quad (17)$$

$$\text{Exact CV} = \int_{p_1^0}^{p_1^1} h_1(p, u^0) dp = A + B + C + D + E, \quad (18)$$

$$\text{Tax Revenue } R = \tau \cdot x^1 = A. \quad (19)$$

Therefore the two DWL measures are:

$$\text{DWL}_M = \text{AV} - R = B + C, \quad (20)$$

$$\text{DWL}_{\text{exact}} = \text{CV} - R = B + C + D + E. \quad (21)$$

5.3 Why Willig's Approximation Works for CV but Fails for DWL

With the area decomposition from Section 5.2, the two approximation errors are:

$$\text{CV} - \text{AV} = (A + B + C + D + E) - (A + B + C) = D + E, \quad (22)$$

$$\text{DWL}_{\text{exact}} - \text{DWL}_M = (B + C + D + E) - (B + C) = D + E. \quad (23)$$

The *absolute* approximation error is identical in both cases. The Marshallian measure understates both CV and the exact DWL by the same amount $D + E$. Nevertheless, the implications for relative accuracy are completely different.

For CV. The relative error is

$$\frac{CV - AV}{CV} = \frac{D + E}{A + B + C + D + E}.$$

The denominator contains the large rectangle $A = \tau \cdot x^1$, which is first-order in τ . The numerator $D + E$ is second-order in τ (it arises from the divergence between the two demand curves, which is proportional to τ , accumulated over a price interval of length τ). Hence the relative CV error is $O(\tau) \rightarrow 0$. This is Willig's result.

For DWL. The relative error is

$$\frac{DWL_{\text{exact}} - DWL_M}{DWL_{\text{exact}}} = \frac{D + E}{B + C + D + E}.$$

Now the rectangle A is *absent* from the denominator. Both numerator and denominator are second-order in τ : $D + E = O(\tau^2)$ and $B + C + D + E = O(\tau^2)$. Therefore the relative DWL error is $O(1)$ —it does not vanish as $\tau \rightarrow 0$.

Remark 5.1 (Hausman's punch line). Willig's bound controls the absolute error $D + E$, which is negligible relative to the first-order quantity CV. But the very same absolute error is *not* negligible relative to the second-order quantity DWL. In the gasoline example: $D + E$ is only 3.2% of CV but 31.7% of the exact DWL. The Marshallian approximation is reliable for CV and unreliable for DWL, even for the same good, at the same tax rate, with the same underlying parameters.

Remark 5.2. In the explanation above, we claimed the decisive contrast in the orders of magnitude between A and $B + C, D + E$:

- A is $O(\tau)$ (a trapezoid of width τ and average height $\sim x^0$).
- $B + C$ and $D + E$ is $O(\tau^2)$ (a triangle of base τ and height proportional to τ).

We make the argumen more formal as follows. Let $\alpha = \partial x_1^* / \partial p_1 < 0$ is the slope of the Marshallian demand curve. Region $A = \tau \cdot x^1$ is a rectangle whose width $x^1 \approx x^0$ is $O(1)$, so $A = O(\tau)$:

Regions $B + C$. Linearizing $x^*(p) \approx x^0 + \alpha(p - p^0)$ gives $x^1 \approx x^0 + \alpha\tau$, so

$$B + C = AV - R = \int_{p^0}^{p^1} x^*(p) dp - \tau \cdot x^1 \approx (x^0\tau + \frac{1}{2}\alpha\tau^2) - (x^0\tau + \alpha\tau^2) = -\frac{1}{2}\alpha\tau^2 = O(\tau^2).$$

Regions $D + E$. Both curves pass through x^0 at p^0 . By the Slutsky equation, $\partial h_1 / \partial p - \partial x_1^* / \partial p = \delta x_1^*$, so the gap between the curves grows as $h(p) - x^*(p) \approx \delta x^0(p - p^0)$. Integrating:

$$D + E = CV - AV = \int_{p^0}^{p^1} [h(p) - x^*(p)] dp \approx \int_0^\tau \delta x^0 s ds = \frac{1}{2}\delta x^0 \tau^2 = O(\tau^2).$$

Since $B + C = O(\tau^2)$ and $D + E = O(\tau^2)$, each of B, C, D, E individually is $O(\tau^2)$.

6 Numerical Examples

6.1 Labor Supply: Large Budget Share

Hausman first considers wives' labor supply (his eq. 35):

$$h_j = 495.1 w_j - 0.1250 y_j + 765.1, \quad (24)$$

where h_j is hours per year, w_j is the net wage (mean \$4.15/hour), and y_j is husband's income (mean \$8,236). In the framework of (5), the parameters are

$$\alpha = 495.1, \quad \delta = -0.125, \quad z\gamma = 765.1, \quad p_1 \rightarrow w.$$

Note that $\delta < 0$ here: a higher husband's income reduces the wife's labor supply (leisure is normal in the consumption sense).

Remark 6.1 (The baseline in the labor supply context). There is a subtlety in applying (9) to labor supply here but you can define leisure demand as

$$x_j \equiv 12 * 365 - h_j$$

assuming 12 hours is the limit of the hours one can work for. Then, we can obtain the natural demand function.

Also another subtlety in the following analysis is that the original "income" is define as the initial *labor earnings* $w^0 \cdot h^0$, not the husband's nonlabor income. This reflects the fact that, in the labor supply context, the wage w plays the role of a price and labor earnings wh is the "expenditure" that must be re-evaluated at the new wage to maintain utility.

I believe it can be fixed by using x_j because the change is just adding a constant to all terms.

A 20% proportional tax on labor earnings reduces the net wage from $w^0 = \$4.15$ to $w^1 = 0.8 \times 4.15 = \3.32 .

Step 1: Initial and post-tax hours.

$$\begin{aligned} h^0 &= 495.1(4.15) - 0.125(8236) + 765.1 = 2054.7 - 1029.5 + 765.1 = 1,790.3 \text{ hrs/yr}, \\ h^1 &= 495.1(3.32) - 0.125(8236) + 765.1 = 1643.7 - 1029.5 + 765.1 = 1,379.3 \text{ hrs/yr}. \end{aligned}$$

Initial annual labor earnings:

$$w^0 \cdot h^0 = 4.15 \times 1790.3 \approx \$7,429.$$

Step 2: Marshallian approximation. The Marshallian welfare measure is the area to the left of the labor supply curve between w^1 and w^0 , holding $y = 8236$ fixed. With y fixed, the labor supply reduces to a function of w alone:

$$h(w, 8236) = 495.1 w + (765.1 - 0.125 \times 8236) = 495.1 w - 264.4.$$

Then:

$$\begin{aligned}
AV &= \int_{w^1}^{w^0} (495.1 w - 264.4) dw = \left[\frac{495.1 w^2}{2} - 264.4 w \right]_{3.32}^{4.15} \\
&= (247.55 \times 4.15^2 - 264.4 \times 4.15) - (247.55 \times 3.32^2 - 264.4 \times 3.32) \\
&= (4263.2 - 1097.3) - (2728.6 - 877.8) \\
&= 3165.9 - 1850.8 = \$1,315.1 \approx \boxed{\$1,315}.
\end{aligned}$$

Step 3: Exact CV via the expenditure function. Substituting the labor supply parameters into (9) (Hausman's eq. 36):

$$\begin{aligned}
e(w, \bar{u}) &= e^{\delta w} \bar{u} - \frac{1}{\delta} \left(\alpha w + \frac{\alpha}{\delta} + z\gamma \right) \\
&= e^{-0.125 w} \bar{u} + 8 (495.1 w - 3960.8 + 765.1) \\
&= e^{-0.125 w} \bar{u} + 3960.8 w - 25,565.6.
\end{aligned}$$

Using (8) to compute $u^0 = v(w^0, y_{\text{labor}}^0)$ where $y_{\text{labor}}^0 = w^0 \cdot h^0 = \$7,429$, and then evaluating $e(w^1, u^0)$, Hausman obtains

$$e(w^1, u^0) = \$9,485 \text{ per year},$$

so that the exact compensating variation is

$$CV = e(w^1, u^0) - y_{\text{labor}}^0 = \$9,485 - \$7,429 = \boxed{\$2,056 \text{ per year}}.$$

Comparison.

Exact CV	\$2,056 per year
Marshallian approximation	\$1,315 per year
Percentage difference	44.6%

Remark 6.2. The approximation is poor because labor income is a large share of the total budget, violating Willig's conditions. In other words, the Willig's bound

$$\frac{CV - AV}{|AV|} \approx \frac{\eta |AV|}{2I} \approx 4.5\%$$

does not work here (which is just 1/10 of the true value) as the assumption that $(1 - \eta)AV/I$ is small. (Willig used the assumption to apply Taylor approximation in a step.)¹

¹The exact bound also does not work here. (I do not know the reason.) Since the exact bound relies on the bounds of true elasticity across relevant price reason, I used the true value of η at the original price.

6.2 Gasoline: Small Budget Share but Large DWL Error

Hausman then considers gasoline demand (his eq. 37):

$$q_j = -14.22p + 0.082y_j + 4.95,$$

with $y = \$720/\text{month}$, $p^0 = \$0.75/\text{gallon}$, price elasticity 0.2, income elasticity 1.1. A tax doubles the price to $p^1 = \$1.50/\text{gallon}$.

	Exact (Hicksian)	Marshallian	% Difference
Compensating variation	\$37.17/month	\$35.99/month	3.2%
Deadweight loss	\$2.88/month	\$3.96/month	31.7%

Note that $DWL_M = \$3.96 > \$2.88 = DWL_{\text{exact}}$, consistent with the analysis in Section 5: the Marshallian (Harberger) triangle overestimates the true deadweight loss for normal goods.

Remark 6.3. In the homework question, you will find that Willig's bound here is

$$\frac{\eta|AV|}{2I} \approx 3\%$$

Remark 6.4. This is Hausman's central result. Even though gasoline is a small part of the budget (Willig's conditions are satisfied, and the CV approximation is excellent at 3.2% error), the deadweight loss estimate is off by nearly a third. In the language of Section 5.2: the absolute error $|E - C|$ is small (only 3.2% of the large CV trapezoid), but it is 31.7% of the much smaller DWL triangle. The error and the DWL are both $O(\tau^2)$, so the percentage error does *not* vanish as $\tau \rightarrow 0$.

7 The Many-Good Case (Hausman's Section II)

In practice, we rarely observe the complete demand system. Hausman addresses this by invoking Hicks' composite commodity theorem.

Proposition 7.1 (Hicks' composite commodity theorem applied to welfare analysis). *Consider N goods with prices $p = (p_1, p_2, \dots, p_N)$ and income y . Suppose only p_1 changes (from p_1^0 to p_1^1) while $p_2, \dots, p_N$ remain fixed. Let $e(p_1, p_2, \dots, p_N, \bar{u})$ be the true expenditure function and let $\tilde{e}(p_1, q, \bar{u})$ be the "quasi" expenditure function obtained by treating goods 2, $\dots$, N as a single composite good with fixed price index q . Then the compensating variation computed from $\tilde{e}$ equals the compensating variation computed from e :*

$$\tilde{e}(p_1^1, q, u^0) - \tilde{e}(p_1^0, q, u^0) = e(p_1^1, p_{-1}^0, u^0) - e(p_1^0, p_{-1}^0, u^0).$$

Proof. Since $p_2, \dots, p_N$ are fixed at $p_{-1}^0 = (p_2^0, \dots, p_N^0)$, we can write them as scalar multiples of a single price index: $p_i^0 = \lambda_i q$ for $i = 2, \dots, N$, where $\lambda_i > 0$ are known constants and $q > 0$ is the price index.

Step 1: The true expenditure function depends on p_{-1} only through q . The true expenditure minimization problem is

$$e(p_1, p_{-1}^0, \bar{u}) = \min_{x \in \mathbb{R}_+^N} \left\{ p_1 x_1 + \sum_{i=2}^N p_i^0 x_i : u(x_1, x_2, \dots, x_N) \geq \bar{u} \right\}.$$

Since $p_i^0 = \lambda_i q$ for $i \geq 2$, this becomes

$$e(p_1, p_{-1}^0, \bar{u}) = \min_{x_1, x_{-1}} \left\{ p_1 x_1 + q \sum_{i=2}^N \lambda_i x_i : u(x_1, x_{-1}) \geq \bar{u} \right\}.$$

Define the composite good $x_c = \sum_{i=2}^N \lambda_i x_i$ with price q . Provided we can optimize over $x_2, \dots, x_N$ conditional on x_c (which we can, since the λ_i ratios are fixed), the problem reduces to

$$\tilde{e}(p_1, q, \bar{u}) = \min_{x_1, x_c} \{ p_1 x_1 + q x_c : \tilde{u}(x_1, x_c) \geq \bar{u} \},$$

where $\tilde{u}(x_1, x_c) = \max\{u(x_1, x_{-1}) : \sum_{i=2}^N \lambda_i x_i = x_c\}$ is the utility function over the two “goods” (x_1, x_c) .

Step 2: The CV depends only on the change in p_1 . The compensating variation from the true expenditure function is

$$\text{CV} = e(p_1^1, p_{-1}^0, u^0) - e(p_1^0, p_{-1}^0, u^0).$$

Since p_{-1}^0 is the same in both terms, and by Step 1 the true expenditure function evaluated at fixed p_{-1}^0 equals the quasi expenditure function evaluated at fixed q , we have

$$e(p_1^1, p_{-1}^0, u^0) = \tilde{e}(p_1^1, q, u^0), \quad e(p_1^0, p_{-1}^0, u^0) = \tilde{e}(p_1^0, q, u^0).$$

Therefore $\text{CV} = \tilde{e}(p_1^1, q, u^0) - \tilde{e}(p_1^0, q, u^0)$.

Step 3: The ODE is the same. By Shephard’s lemma applied to both the true and quasi expenditure functions:

$$\frac{\partial e(p_1, p_{-1}^0, \bar{u})}{\partial p_1} = h_1(p_1, p_{-1}^0, \bar{u}) = x_1^*(p_1, p_{-1}^0, e) \quad \text{and} \quad \frac{\partial \tilde{e}(p_1, q, \bar{u})}{\partial p_1} = \tilde{h}_1(p_1, q, \bar{u}) = x_1^*(p_1, q, \tilde{e}).$$

Since $e = \tilde{e}$ when $p_{-1} = p_{-1}^0$ (Step 2) and the Marshallian demand x_1^* depends on other prices only through q (because p_{-1}^0 is fixed at $\lambda_i q$), both functions satisfy the *same* ODE

$$\frac{d\tilde{e}}{dp_1} = x_1^*(p_1, q, \tilde{e})$$

with the same initial condition $\tilde{e}(p_1^0, q, u^0) = y^0$. By uniqueness of ODE solutions, $\tilde{e}(p_1, q, \bar{u}) = e(p_1, p_{-1}^0, \bar{u})$ for all p_1 . $\square$

Remark 7.2. The key point is that when only one price changes, we do not need to estimate the complete demand system or recover the full expenditure function $e(p_1, \dots, p_N, \bar{u})$. The quasi expenditure function $\tilde{e}(p_1, q, \bar{u})$ —which requires only the single demand equation x_1^* —gives the exact same welfare measure. This is what justifies Hausman’s entire approach.

This result justifies the entire approach: to compute exact welfare measures for a single price change, one needs only the estimated demand equation for the good whose price changes. No information about the rest of the demand system is required.

8 Summary

The Hausman (1981) procedure for exact welfare analysis:

1. **Estimate** a single Marshallian demand function $x_1^*(p_1, y)$ from data.
2. **Formulate the ODE**: $dy/dp_1 = x_1^*(p_1, y)$ along an indifference curve.
3. **Solve the ODE** (e.g. via integrating factor for linear demand) to obtain the indirect utility function and expenditure function:
 - Linear demand: closed-form solution (9).
 - Log-linear demand: closed-form solution (14).
 - General demand: numerical integration (e.g., Runge–Kutta).
4. **Compute exact CV**: $CV = e(p_1^1, u^0) - I$.
5. **Compute exact DWL**: $DWL_{\text{exact}} = CV - \tau \cdot h_1(p_1^1, u^0)$, using the *Hicksian* quantity at p^1 .
6. **Compute standard errors** via the delta method applied to the closed-form expressions.

References

- Diamond, P. and D. McFadden (1974). “Some Uses of the Expenditure Function in Public Finance.” *J. Public Econ.*, 3, 3–21.
- Diewert, W. E. (1974). “Applications of Duality Theory.” In *Frontiers of Quantitative Economics*, Vol. II, ed. M. D. Intriligator and D. A. Kendrick. Amsterdam: North-Holland.
- Hausman, J. A. (1981). “Exact Consumer’s Surplus and Deadweight Loss.” *American Economic Review*, 71(4), 662–676.
- Hurwicz, L. and H. Uzawa (1971). “On the Integrability of Demand Functions.” In *Preferences, Utility and Demand*, ed. J. S. Chipman et al. New York: Harcourt Brace Jovanovich.
- McKenzie, L. W. (1957). “Demand Theory Without a Utility Index.” *Rev. Econ. Stud.*, 24, 185–189.
- Samuelson, P. A. (1950). “The Problem of Integrability in Utility Theory.” *Economica*, 17, 355–385.
- Varian, H. (1978). *Microeconomic Analysis*. New York: W. W. Norton.
- Willig, R. D. (1976). “Consumer’s Surplus Without Apology.” *American Economic Review*, 66(4), 589–597.