

Lecture Note on Welfare Analysis

Kota Saito

Contents

1 Preliminaries	2
1.1 The Expenditure Minimization Problem	2
1.2 The Indirect Utility Function and Roy's Identity	3
1.3 Duality of the Consumer's Problem	4
1.4 The Slutsky Equation	4
2 Welfare Evaluation of Price Changes	5
2.1 A Money Metric: The Indirect Utility Function	6
2.2 Compensating Variation	6
2.3 Equivalent Variation	7
2.4 The Area Variation (Marshallian Consumer Surplus)	7
2.5 The Ordering: CV, AV, and EV	8
A Integrability Condition and Slutsky Symmetry	9
A.1 Exact Differential Forms and Conservative Vector Fields	10
A.2 Constructing e by Integration	11
A.3 Why Symmetry Guarantees Path Independence	11
A.4 What Goes Wrong Without Symmetry	11
B Path-Dependence of Consumer Surplus with Multiple Price Changes	12
B.1 The Problem	12
B.2 Subsequent Literature	12
B.3 Implications for Applied Welfare Analysis	14
C Income Effect: Why CV and EV can be different?	15
C.1 When Is the Income Effect Large or Small?	15

1 Preliminaries

Throughout this note, we consider a consumer with a continuous, strictly quasi-concave, and locally nonsatiated utility function $u : \mathbb{R}_+^N \rightarrow \mathbb{R}$. The consumer faces prices $p = (p_1, \dots, p_N) \in \mathbb{R}_{++}^N$ and income $I > 0$.

1.1 The Expenditure Minimization Problem

Definition 1.1 (Compensated demand and expenditure function). Given prices $p \in \mathbb{R}_{++}^N$ and a target utility level $\bar{u}$, define the **expenditure function**

$$e(p, \bar{u}) \equiv \min_{x \in \mathbb{R}_+^N} p \cdot x \quad \text{subject to} \quad u(x) \geq \bar{u},$$

and the **compensated (Hicksian) demand function**

$$\bar{x}(p, \bar{u}) \equiv \arg \min_{x \in \mathbb{R}_+^N} \{p \cdot x : u(x) \geq \bar{u}\}.$$

The expenditure function and compensated demand are linked by $e(p, \bar{u}) = p \cdot \bar{x}(p, \bar{u})$.

The compensated demand $\bar{x}(p, \bar{u})$ answers: “what is the cheapest bundle that achieves utility $\bar{u}$ at prices p ?”

Proposition 1.2 (Properties of the expenditure function). *Under the maintained assumptions on u , the expenditure function satisfies:*

- (a) **Non-decreasing in p :** $p' \geq p \implies e(p', \bar{u}) \geq e(p, \bar{u})$.
- (b) **Homogeneous of degree 1 in p :** $e(tp, \bar{u}) = t e(p, \bar{u})$ for all $t > 0$.
- (c) **Concave in p :** For all $\lambda \in [0, 1]$,

$$e(\lambda p + (1 - \lambda)p', \bar{u}) \geq \lambda e(p, \bar{u}) + (1 - \lambda) e(p', \bar{u}).$$

(d) **Shephard's lemma:** $\frac{\partial e(p, \bar{u})}{\partial p_i} = \bar{x}_i(p, \bar{u})$.

- (e) **Strictly increasing in $\bar{u}$** (provided utility is locally non-satiated).

Proof. Statements (a), (b), and (e) follow directly from the definition.

For (c), fix $\lambda \in [0, 1]$. Let $x^*(\tilde{p}, \bar{u})$ denote the solution of the expenditure minimization problem at prices $\tilde{p}$. Then

$$\begin{aligned} e(\lambda p + (1 - \lambda)p', \bar{u}) &= (\lambda p + (1 - \lambda)p') \cdot x^*(\lambda p + (1 - \lambda)p', \bar{u}) \\ &\geq \lambda(p \cdot x^*(p, \bar{u})) + (1 - \lambda)(p' \cdot x^*(p', \bar{u})) \\ &= \lambda e(p, \bar{u}) + (1 - \lambda) e(p', \bar{u}), \end{aligned}$$

where the inequality uses the fact that $x^*(\tilde{p}, \bar{u})$ minimizes cost at prices $\tilde{p}$, so $\tilde{p} \cdot x^*(\tilde{p}, \bar{u}) \leq \tilde{p} \cdot x$ for any feasible x .

For (d), this follows from the envelope theorem applied to the EMP. The Lagrangian is $\mathcal{L} = p \cdot x - \mu(u(x) - \bar{u})$. By the envelope theorem,

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} \Big|_{x=\bar{x}} = \bar{x}_i(p, \bar{u}). \quad \square$$

Remark 1.3. The expenditure function measures the minimum cost of attaining utility $\bar{u}$ at prices p . It is the key tool for welfare analysis: the change $e(p', \bar{u}) - e(p, \bar{u})$ quantifies the monetary impact of a price change from p to p' , holding welfare constant.

1.2 The Indirect Utility Function and Roy's Identity

Definition 1.4 (Indirect utility function). The **indirect utility function** is the value function of the utility maximization problem (UMP):

$$v(p, I) \equiv \max_{x \geq 0} u(x) \quad \text{subject to} \quad p \cdot x \leq I.$$

It records the maximized utility level attainable at prices $p \in \mathbb{R}_{++}^N$ and income $I > 0$.

Definition. A function $f : S \rightarrow \mathbb{R}$ defined on a convex set S is *quasi-convex* if for every $\bar{v} \in \mathbb{R}$, the lower contour set

$$L(\bar{v}) = \{x \in S : f(x) \leq \bar{v}\}$$

is convex. Equivalently, for all $x, y \in S$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.$$

Proposition 1.5 (Properties of $v(p, I)$). *Suppose u is continuous and locally nonsatiated. Then $v(p, I)$ is:*

- (i) *Homogeneous of degree zero in (p, I) : $v(\lambda p, \lambda I) = v(p, I)$ for all $\lambda > 0$.*
- (ii) *Nondecreasing in I and nonincreasing in each p_i .*
- (iii) *Quasiconvex in (p, I) .*
- (iv) *Continuous in (p, I) .*

Proof sketch. (i) Multiplying all prices and income by $\lambda > 0$ leaves the budget set unchanged. (ii) A higher income relaxes the constraint; a higher p_i tightens it. (iv) By the Maximum Theorem: the budget correspondence is continuous and u is continuous, so v is continuous.

We now prove (iii) in the following.

Goal: Show that $\{(p, I) : v(p, I) \leq \bar{v}\}$ is convex for every $\bar{v}$.

Step 1. Fix any bundle $\bar{x}$. The set $\{(p, I) : p \cdot \bar{x} > I\}$ —where $\bar{x}$ is *unaffordable*—is a half-space, hence convex.

Step 2. We characterize the lower contour set. The condition $v(p, I) \leq \bar{v}$ means that no bundle with utility above $\bar{v}$ is affordable. Therefore:

$$\{(p, I) : v(p, I) \leq \bar{v}\} = \bigcap_{x: u(x) > \bar{v}} \{(p, I) : p \cdot x > I\}.$$

Step 3. Each set in the intersection is convex (Step 1). An arbitrary intersection of convex sets is convex.

Conclusion. The lower contour set is convex, so v is quasi-convex. $\square$

Proposition 1.6 (Roy's Identity). *Suppose $v(p, I)$ is differentiable and $\frac{\partial v}{\partial I}(p, I) \neq 0$. Then the Marshallian demand for good i satisfies*

$$x_i^*(p, I) = -\frac{\partial v(p, I)/\partial p_i}{\partial v(p, I)/\partial I}.$$

Proof. The Lagrangian for the UMP is $\mathcal{L} = u(x) - \lambda(p \cdot x - I)$, with optimal multiplier λ^* . By the envelope theorem:

$$\frac{\partial v}{\partial p_i} = -\lambda^* x_i^*, \quad \frac{\partial v}{\partial I} = \lambda^*.$$

Dividing gives the result. $\square$

Remark 1.7. Roy's identity allows one to recover the entire demand system from the single scalar function v . Intuitively, a one-unit rise in p_i reduces utility by $\lambda^* x_i^*$ (it costs x_i^* extra units of income, each worth λ^* in utility). Dividing by λ^* converts this utility loss back into income units, yielding x_i^* .

1.3 Duality of the Consumer's Problem

Proposition 1.8 (Duality). *Let x^* be the optimal bundle at (p, I) , and let $\bar{u} = v(p, I)$. Then:*

1. $x(p, I) = x^* = x^h(p, \bar{u})$ (Marshallian = Hicksian at the optimum);
2. $e(p, v(p, I)) = I$ (expenditure recovers income);
3. $v(p, e(p, \bar{u})) = \bar{u}$ (indirect utility recovers the target utility).

Remark 1.9. Whether we frame the consumer's problem as "maximize utility given a budget" or "minimize cost to achieve a target utility," we arrive at the same optimal bundle. This duality is the engine behind the Slutsky decomposition and welfare analysis.

1.4 The Slutsky Equation

Theorem 1.10 (Slutsky Equation). *Let $\bar{u} = v(p, I)$. Then for each good i and price p_i ,*

$$\frac{\partial x_i(p, I)}{\partial p_i} = \underbrace{\frac{\partial x_i^h(p, \bar{u})}{\partial p_i}}_{\text{Substitution Effect (SE)}} - \underbrace{\frac{\partial x_i(p, I)}{\partial I} x_i^*}_{\text{Income Effect (IE)}}.$$

Proof. By duality (Proposition 1.8), for all p and corresponding $\bar{u} = v(p, I)$:

$$x_i^h(p, \bar{u}) = x_i(p, e(p, \bar{u})).$$

Differentiating both sides with respect to p_i (applying the chain rule on the right-hand side):

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, e(p, \bar{u}))}{\partial p_i} + \frac{\partial x_i(p, e(p, \bar{u}))}{\partial I} \cdot \frac{\partial e(p, \bar{u})}{\partial p_i}.$$

By Shephard's lemma, $\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}) = x_i^*$. Substituting and noting $e(p, \bar{u}) = I$:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, I)}{\partial p_i} + \frac{\partial x_i(p, I)}{\partial I} x_i^*.$$

Rearranging yields the Slutsky equation. □

The total effect of an own-price increase on Marshallian demand decomposes as:

$$\underbrace{\frac{\partial x_i(p, I)}{\partial p_i}}_{\text{Total Effect}} = \underbrace{\frac{\partial x_i^h(p, \bar{u})}{\partial p_i}}_{\text{SE: } \leq 0} - \underbrace{\frac{\partial x_i(p, I)}{\partial I} \cdot x_i^*}_{\text{IE}}.$$

- **Substitution Effect (SE):** The change in demand holding utility constant (movement along the indifference curve). By the negativity of the self-substitution effect, $\frac{\partial x_i^h}{\partial p_i} \leq 0$ always.
- **Income Effect (IE):** The change in demand due to the reduction in real income caused by the price increase, multiplied by the original quantity consumed.
 - If good i is a *normal good* ($\partial x_i / \partial I \geq 0$): the IE is negative (reinforces the SE).
 - If good i is an *inferior good* ($\partial x_i / \partial I < 0$): the IE is positive (offsets the SE).
 - If good i is a *Giffen good*: the IE dominates the SE, so $\partial x_i / \partial p_i > 0$.

Remark 1.11. The general (cross-price) Slutsky equation for the effect of p_j on x_i takes the form

$$\frac{\partial x_i(p, I)}{\partial p_j} = \frac{\partial x_i^h(p, \bar{u})}{\partial p_j} - \frac{\partial x_i(p, I)}{\partial I} x_j^*.$$

2 Welfare Evaluation of Price Changes

In this section we study how to measure, in monetary terms, the welfare gain or loss a consumer experiences when prices change. We develop three measures—the compensating variation (CV), the equivalent variation (EV), and the area variation (AV, also called the change in Marshallian consumer surplus)—and establish their relationship.

Throughout, we consider a consumer facing an initial price vector p^0 and a new price vector p^1 , with income I held fixed. For notational simplicity, we focus on a change in the price of good 1 alone: $p_1^0 = a$ and $p_1^1 = b$ with $b > a$ (a price increase), while $p_{-1}^0 = p_{-1}^1 = \bar{p}_{-1}$.

Example 2.1 (Electricity price increase). An earthquake damages a power plant, forcing an electricity company to raise the price of power from a to b . The government orders the company to compensate consumers. How much compensation is required to leave consumers no worse off than before?

Denote by $u^0 = v(p^0, I)$ and $u^1 = v(p^1, I)$ the utility levels before and after the price change, where v is the indirect utility function. Then $u^1 < u^0$.

2.1 A Money Metric: The Indirect Utility Function

Given any price vector p and any indirect utility function $v(p, I)$, we know that v is strictly increasing in I (by local nonsatiation). Hence, for a fixed reference price vector $\bar{p}$, the function

$$e(\bar{p}, v(p, I))$$

is a *money metric indirect utility function*: it measures utility in dollar units at reference prices $\bar{p}$. It is strictly increasing in I and provides a valid indirect utility function representing $\succsim$.

If $v(p', I) > v(p^0, I)$, then $e(\bar{p}, v(p', I)) > e(\bar{p}, v(p^0, I))$: the consumer is better off, and the difference is expressed in dollars. Both CV and EV are instances of this idea, differing only in the choice of reference prices.

2.2 Compensating Variation

Definition 2.2 (Compensating Variation). The **compensating variation** (CV) is the amount of income the consumer must receive after the price change to restore the original utility level u^0 :

$$CV \equiv e(p^1, u^0) - e(p^0, u^0).$$

Since $e(p^0, u^0) = I$, this simplifies to $CV = e(p^1, u^0) - I$.

Remark 2.3. To see the similarity with money metric utility, notice that CV can be written as

$$CV \equiv e(p^1, u^0) - e(p^0, u^0) = e(p^1, u^0) - e(p^1, u^1).$$

thus CV is the difference of money metric utility with $\bar{p} = p^1$.

Interpretation. The CV answers: “After the price change occurs, how much additional income would the consumer need to be just as well off as before?” Equivalently, $CV = e(p^1, u^0) - e(p^1, u^1)$, which is the difference between the cost of achieving u^0 and the cost of achieving u^1 at the new prices—both evaluated at p^1 .

For a price increase ($b > a$), the consumer is worse off, so $CV > 0$: the consumer needs a positive transfer to be compensated.

CV as area under compensated demand. By Shephard’s lemma, $\partial e(p, u^0)/\partial p_1 = x_1^h(p, u^0)$, so:

$$CV = e(p^1, u^0) - e(p^0, u^0) = \int_{p_1^0}^{p_1^1} x_1^h(p_1, \bar{p}_{-1}, u^0) dp_1. \quad (1)$$

The CV is the area to the left of the initial Hicksian demand curve $x_1^h(\cdot, u^0)$ between p_1^0 and p_1^1 .

2.3 Equivalent Variation

Definition 2.4 (Equivalent Variation). The **equivalent variation** (EV) is the amount of income the consumer would be willing to pay, before the price change, to avoid it:

$$\text{EV} \equiv e(p^1, u^1) - e(p^0, u^1).$$

Since $e(p^1, u^1) = I$, this simplifies to $\text{EV} = I - e(p^0, u^1)$.

Remark 2.5. To see the similarity with money metric utility, notice that EV can be written as

$$\text{EV} \equiv e(p^1, u^1) - e(p^0, u^1) = e(p^0, u^0) - e(p^0, u^1).$$

thus EV is the difference of money metric utility with $\bar{p} = p^0$.

Interpretation. From the definition, $\text{EV} = e(p^1, u^1) - e(p^0, u^1)$. Since $e(p^1, u^1) = I$ (by duality), we obtain

$$\text{EV} = I - e(p^0, u^1), \quad \text{i.e.,} \quad e(p^0, u^1) = I - \text{EV}.$$

Applying the duality relation $v(p, e(p, \bar{u})) = \bar{u}$ (Proposition 1.8) with $p = p^0$ and $\bar{u} = u^1$ gives

$$v(p^0, I - \text{EV}) = v(p^0, e(p^0, u^1)) = u^1.$$

That is, if prices remain at p^0 but income is reduced by EV, the consumer attains exactly the post-change utility level u^1 . The consumer is therefore indifferent between two scenarios:

1. Prices rise to p^1 , income stays at I .
2. Prices stay at p^0 , income falls to $I - \text{EV}$.

In this sense, the income loss of EV dollars at old prices is **welfare-equivalent** to the price change—hence the name. Equivalently, the consumer would be willing to pay up to EV at the original prices to prevent the price change.

EV as area under compensated demand. By Shephard's lemma applied at utility level u^1 :

$$\text{EV} = e(p^1, u^1) - e(p^0, u^1) = \int_{p_1^0}^{p_1^1} x_1^h(p_1, \bar{p}_{-1}, u^1) dp_1. \quad (2)$$

The EV is the area to the left of the final Hicksian demand curve $x_1^h(\cdot, u^1)$ between p_1^0 and p_1^1 .

2.4 The Area Variation (Marshallian Consumer Surplus)

A practical difficulty with both CV and EV is that the Hicksian demand functions $x_1^h(\cdot, u^0)$ and $x_1^h(\cdot, u^1)$ are not directly observable. Only the Marshallian demand $x_1(p, I)$ can be estimated from data. This motivates a third measure.

Definition 2.6 (Area Variation). The **area variation** (AV), also called the change in Marshallian consumer surplus, is

$$AV = \int_{p_1^0}^{p_1^1} x_1(p_1, \bar{p}_{-1}, I) dp_1. \quad (3)$$

It is the area to the left of the Marshallian (ordinary) demand curve between $p_1 = a$ and $p_1 = b$.

The AV is directly computable from observable demand data and does not require knowledge of the consumer's preferences.

Remark 2.7. AV is often referred to as consumer surplus, largely for historical reasons tracing back to Marshall's terminology. Subsequent work questioned the welfare interpretation of this measure and introduced EV and CV as theoretically grounded alternatives. AV is best viewed as a practical approximation to EV and CV—an interpretation formalized by Willig (1976).

AV may not be uniquely defined when multiple prices change simultaneously. See Appendix B.

2.5 The Ordering: CV, AV, and EV

The three measures are related through the Slutsky equation. The key observation is that, for a price increase of a normal good, the Hicksian demand curve at utility u^0 lies to the right of the Marshallian demand curve, which in turn lies to the right of the Hicksian demand curve at u^1 .

Proposition 2.8 (Ordering of welfare measures for normal goods). *Suppose good 1 is a normal good ($\partial x_1 / \partial I > 0$). Then*

(i) for the price increase $p_1^1 \geq p_1^0$:

$$CV \geq AV \geq EV > 0;$$

(ii) for the price decrease $p_1^1 \leq p_1^0$:

$$0 > CV \geq AV \geq EV.$$

For the case of inferior goods ($\partial x_1 / \partial I < 0$), the positions of CV and EV are flipped and the other parts are the same.

Proof. Consider the normal goods case and price increase case. The Slutsky equation gives, for each p_1 ,

$$\frac{\partial x_1^h(p_1, u)}{\partial p_1} = \frac{\partial x_1(p_1, I)}{\partial p_1} + \frac{\partial x_1(p_1, I)}{\partial I} x_1.$$

Since the income effect $(\partial x_1 / \partial I) x_1 > 0$ for a normal good, the Hicksian demand curve is steeper (less elastic) than the Marshallian demand curve. Moreover, since $u^0 > u^1$ (the price increase lowers utility) and good 1 is normal, we have

$$x_1^h(p_1, u^0) \geq x_1(p_1, I) \geq x_1^h(p_1, u^1) \quad \text{for all } p_1 \in [p_1^0, p_1^1].$$

Integrating from p_1^0 to p_1^1 yields $CV \geq AV \geq EV > 0$.

For the price decrease case, we have $u^0 < u^1$ and thus we have

$$x_1^h(p_1, u^0) \leq x_1(p_1, I) \leq x_1^h(p_1, u^1) \quad \text{for all } p_1 \in [p_1^1, p_1^0].$$

Integrating from p_1^0 to p_1^1 yields the result. Notice that since $p_1^0 > p_1^1$, the values are negative so the order will be flipped. $\square$

Remark 2.9 (When all three coincide). If the utility function is quasi-linear in a numeraire good, i.e. $u(x_1, \dots, x_{N-1}, x_N) = \varphi(x_1, \dots, x_{N-1}) + x_N$, then the income effect for goods $1, \dots, N-1$ is identically zero. In this case, the Hicksian and Marshallian demand curves coincide, and

$$CV = AV = EV.$$

More generally, when the income effect is small (small budget share $w_1 = p_1 x_1 / I$), all three measures are approximately equal, and AV is a good approximation to both CV and EV.

Remark 2.10. The fact that CV and EV are different can be explained by the income effects as follows. Because of income effects, under a lower price, you feel richer (your utility becomes higher) and then, your evaluation of welfare change in terms of monetary amount becomes *larger*. (When you are rich, you do not evaluate marginal value of \$1 much. Thus to evaluate your welfare change in monetary amount, the amount becomes larger.)

- Price increase case: $u^0 > u^1$. Evaluation of welfare change at higher utility u_0 leads to larger monetary value thus $|CV| \geq |EV|$. Both are positive thus, $CV \geq EV \geq 0$.
- Price decrease case: $u^0 < u^1$. Evaluation of welfare change at higher utility u_1 leads to larger monetary value thus $|EV| \geq |CV|$. Both are negative thus $0 \geq CV \geq EV$.

A Integrability Condition and Slutsky Symmetry

The question is as follows: does there exist a scalar function $e : \mathbb{R}_{++}^L \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^h(p, \bar{u}) \quad \text{for all } i = 1, \dots, L?$$

In the language of multivariable calculus, this asks: is the vector field $h = (h_1, \dots, h_L)$ the gradient of some scalar function? Equivalently: is the differential 1-form

$$\omega = \sum_{i=1}^L x_i^h(p, \bar{u}) dp_i$$

exact?

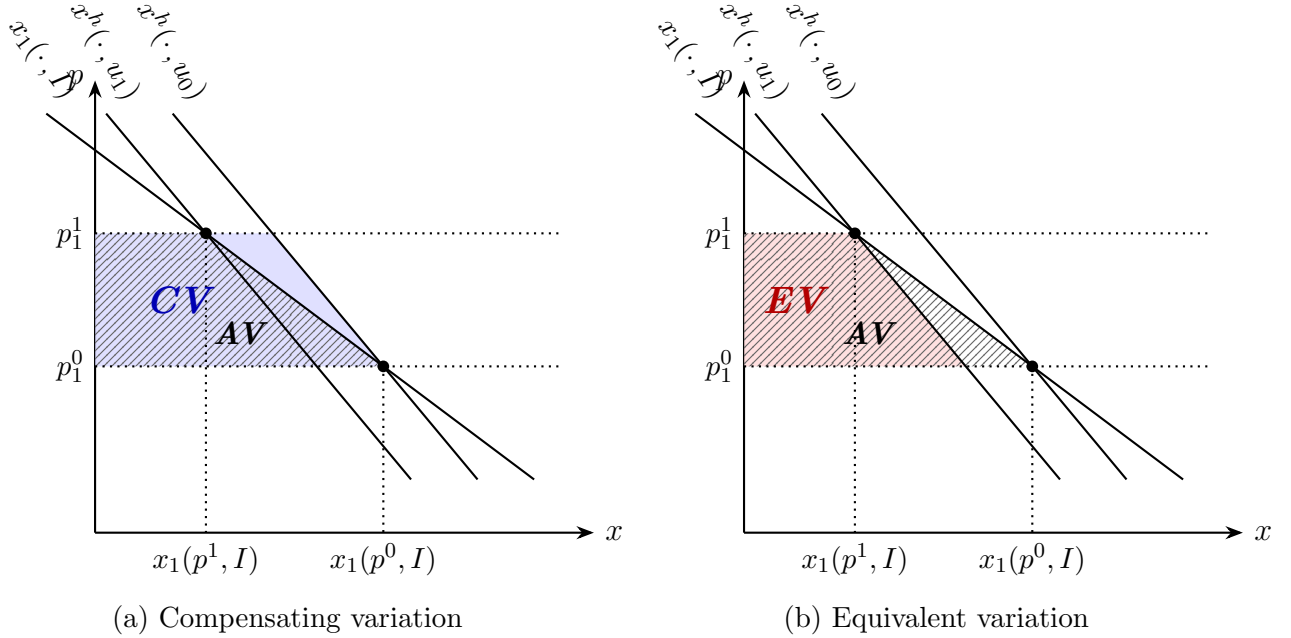


Figure 1: EV and CV from Hicksian demand curves. Price increases from p_1^0 to p_1^1 , decreasing utility from u_0 to u_1 . The two steeper lines are the Hicksian demands $x^h(\cdot, u_0)$ and $x^h(\cdot, u_1)$; the less steep line is the Marshallian demand $x_1(\cdot, I)$ as we are assuming normal goods. Panel (a): EV (blue) is the area to the left of $x^h(\cdot, u_1)$; AV (dashed) is the area to the left of $x_1(\cdot, I)$. Panel (b): CV (red) is the area to the left of $x^h(\cdot, u_0)$; AV (dashed) is the area to the left of $x_1(\cdot, I)$. For a normal good with a price increase, $CV > AV > EV$.

A.1 Exact Differential Forms and Conservative Vector Fields

Theorem A.1 (Poincaré Lemma / Exactness Criterion). *Let $\Omega \subseteq \mathbb{R}^L$ be an open, simply connected domain, and let $f_1, \dots, f_L : \Omega \rightarrow \mathbb{R}$ be C^1 functions. The following are equivalent:*

- (a) **Exactness.** *There exists a C^2 function $F : \Omega \rightarrow \mathbb{R}$ such that $\frac{\partial F}{\partial p_i} = f_i(p)$ for all i .*
- (b) **Path independence.** *For any two points $p^0, p^1 \in \Omega$ and any piecewise- C^1 path γ from p^0 to p^1 ,*

$$\int_{\gamma} \sum_i f_i(p) dp_i$$

depends only on the endpoints, not on the path γ .

- (c) **Symmetry of the Jacobian.** *For all $i, j \in \{1, \dots, L\}$,*

$$\frac{\partial f_i}{\partial p_j} = \frac{\partial f_j}{\partial p_i}.$$

Application to Consumer Theory. Set $f_i(p) = x_i^h(p, \bar{u})$ and $F(p) = e(p, \bar{u})$. The domain $\Omega = \mathbb{R}_{++}^L$ is open and simply connected. The theorem says:

The expenditure function $e(p, \bar{u})$ satisfying $\nabla_p e = h(\cdot, \bar{u})$ exists $\iff s_{ij} = s_{ji}$ for all i, j ,

which is Slutsky symmetry.

A.2 Constructing e by Integration

When symmetry holds, the expenditure function is recovered by a line integral. Fix a reference price p^0 and a reference utility level $\bar{u}$ with $e(p^0, \bar{u})$ known. For any target price p , choose any piecewise- C^1 path γ from p^0 to p in $\mathbb{R}_{++}^L$ and define:

$$e(p, \bar{u}) = e(p^0, \bar{u}) + \int_{\gamma} \sum_{i=1}^L x_i^h(q, \bar{u}) dq_i.$$

Slutsky symmetry guarantees this integral is path-independent, so e is well-defined regardless of which path γ is chosen.

A.3 Why Symmetry Guarantees Path Independence

Consider two paths γ_1 and γ_2 from p^0 to p , and let $\Gamma = \gamma_1 - \gamma_2$ denote the closed loop. We need to show $\oint_{\Gamma} \sum_i x_i^h dq_i = 0$.

By Stokes' theorem (Green's theorem when $L = 2$), this loop integral equals a surface integral of the “curl” of h . In two dimensions:

$$\oint_{\Gamma} h_1 dq_1 + h_2 dq_2 = \iint_D \left(\frac{\partial h_2}{\partial q_1} - \frac{\partial h_1}{\partial q_2} \right) dq_1 dq_2.$$

If Slutsky symmetry holds ($\partial h_1 / \partial q_2 = \partial h_2 / \partial q_1$), the integrand vanishes identically. The argument extends to $L \geq 3$ via the general Stokes' theorem: the exterior derivative of $\omega = \sum_i x_i^h dp_i$ is

$$d\omega = \sum_{i < j} \left(\frac{\partial h_j}{\partial p_i} - \frac{\partial h_i}{\partial p_j} \right) dp_i \wedge dp_j.$$

Slutsky symmetry makes every coefficient zero, so $d\omega = 0$ (closed). On the simply connected domain $\mathbb{R}_{++}^L$, closed $\Leftrightarrow$ exact (Poincaré lemma), so $\omega = de$ for some function e .

A.4 What Goes Wrong Without Symmetry

If Slutsky symmetry fails—say $s_{12} \neq s_{21}$ —then different paths of integration from p^0 to p yield different values. No single-valued function $e(p, \bar{u})$ can simultaneously satisfy $\nabla_p e = h$, because a gradient field must have a symmetric Jacobian (Young's theorem). In economic terms: the demand system cannot have arisen from any well-behaved utility function, because the expenditure function that would rationalise it simply does not exist.

B Path-Dependence of Consumer Surplus with Multiple Price Changes

B.1 The Problem

When only a single price changes, the Marshallian consumer surplus $\Delta S = - \int_a^b x_1(p_1, \bar{p}_{-1}, I) dp_1$ is an ordinary one-dimensional integral and is unambiguous.

However, when multiple prices change simultaneously, the surplus becomes a line integral in price space, and its value may depend on the path of integration.

Auerbach's two-market illustration. Consider two goods. If we change p_1 first and then p_2 , the surplus change is

$$\Delta S_1 = - \int_{p_1^0}^{p_1^1} x_1(p_1, p_2^0) dp_1 - \int_{p_2^0}^{p_2^1} x_2(p_1^1, p_2) dp_2.$$

If instead we change p_2 first and then p_1 :

$$\Delta S_2 = - \int_{p_1^0}^{p_1^1} x_1(p_1, p_2^1) dp_1 - \int_{p_2^0}^{p_2^1} x_2(p_1^0, p_2) dp_2.$$

The difference is

$$\begin{aligned} \Delta S_2 - \Delta S_1 &= - \int_{p_1^0}^{p_1^1} [x_1(p_1^1, p_2^1) - x_1(p_1^1, p_2^0)] dp_1 + \int_{p_2^0}^{p_2^1} [x_2(p_1^1, p_2) - x_2(p_1^0, p_2)] dp_2 \\ &= - \int_{p_1^0}^{p_1^1} \left[\frac{\partial x^1}{\partial p_2} dp_2 \right] dp_1 + \int_{p_2^0}^{p_2^1} \left[\frac{\partial x^2}{\partial p_1} dp_1 \right] dp_2, \end{aligned}$$

where $dp^1 = p_1^1 - p_1^0$ and $dp^2 = p_2^1 - p_2^0$.

For infinitesimal changes, $\frac{\partial x^1}{\partial p_2}$ is constant in p^1 ; $\frac{\partial x^2}{\partial p_1}$ is constant in p^2 . Thus it reduces to

$$\Delta S_2 - \Delta S_1 = \left(\frac{\partial x^2}{\partial p_1} - \frac{\partial x^1}{\partial p_2} \right) dp^1 dp^2,$$

which vanishes if and only if $\partial x_1 / \partial p_2 = \partial x_2 / \partial p_1$ —the exactness condition.

B.2 Subsequent Literature

Willig (1976). Showed that for a single price change, the percentage error from using Marshallian surplus in place of CV or EV is bounded by a simple function of the income elasticity η and the surplus-to-income ratio $\Delta S/I$. Specifically:

$$\frac{|\text{CV} - \Delta S|}{|\Delta S|} \leq \frac{|\eta|}{2} \cdot \frac{|\Delta S|}{I}.$$

For goods with small budget shares and moderate income elasticities, this error is negligible.

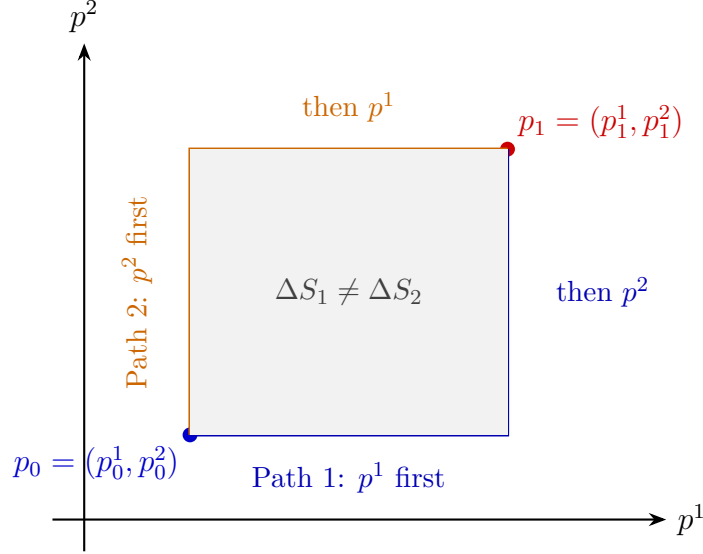


Figure 2: Path-dependence of Marshallian consumer surplus. The line integral $\Delta S = - \int \sum x^i dp^i$ from p_0 to p_1 depends on whether we change p^1 first (Path 1) or p^2 first (Path 2). The two values differ whenever $\partial x^1 / \partial p^2 \neq \partial x^2 / \partial p^1$.

Hausman (1981). Showed how to recover the exact Hicksian demand (and hence exact CV) from an estimated Marshallian demand system by integrating the expenditure function using Roy’s identity and the Slutsky equation. The “exact consumer surplus” approach became standard in applied industrial organisation and public economics.

Chipman and Moore (1976, 1980). Proved a sharp impossibility result:

Proposition B.1 (Chipman–Moore). *Consumer surplus provides a welfare ordering consistent with utility across all possible multi-price changes if and only if preferences admit a quasi-linear representation (zero income effects for all priced goods).*

McKenzie and Pearce (1976). Independently established that Marshallian surplus is path-independent if and only if the Marshallian cross-price Jacobian is symmetric, i.e., $\partial x_i / \partial p_j = \partial x_j / \partial p_i$ for all i, j . By the Slutsky equation and Slutsky symmetry of the Hicksian terms, this condition reduces to

$$\frac{\partial x_i}{\partial I} x_j = \frac{\partial x_j}{\partial I} x_i \quad \text{for all } i, j,$$

which, for strictly positive demands, is equivalent to

$$\frac{\partial x_i / \partial I}{x_i} = \frac{\partial x_j / \partial I}{x_j} \quad \text{for all } i, j.$$

That is, all goods must have the same income elasticity. Differentiating the budget constraint $\sum_k p_k x_k = I$ with respect to I shows that this common elasticity must equal one, so that $x_i(p, I) = I \cdot g_i(p)$ for some functions g_i . This is precisely the condition that preferences are

homothetic. (The quasi-linear case—zero income effects—is a special case in which path independence holds trivially.)

Vives (1987). Extended the analysis to show that consumer surplus remains a valid local approximation for small price changes, even with multiple prices. The path-dependence error is second-order in the size of the price change.

B.3 Implications for Applied Welfare Analysis

Scenario	Recommended measure	Justification
Single price change, small budget share	AV (Marshallian surplus)	Willig bounds: error is negligible
Single price change, exact welfare	CV or EV via Hausman method	Exact; no approximation error
Multiple price changes	CV or EV only	AV is path-dependent (Chipman–Moore)
Multiple small price changes	AV acceptable as approximation	Path-dependence is second-order (Vives)
Quasi-linear preferences	$AV = CV = EV$	All three coincide; no path-dependence

The overarching conclusion: modern applied welfare economics predominantly uses CV and EV rather than Marshallian surplus, especially when multiple prices change or when precision matters. The Marshallian surplus remains a useful shortcut for single-market, small-change analysis, but its theoretical foundations are limited.

C Income Effect: Why CV and EV can be different?

When the price of a good changes, two things happen simultaneously. First, relative prices shift, inducing substitution toward the now-cheaper good—this is the **substitution effect**. Second, the consumer’s real purchasing power changes: a price decrease makes you effectively richer, and a price increase makes you effectively poorer. This change in behavior *due solely to the change in real wealth* is the **income effect**.

Formally, the Slutsky decomposition tells us:

$$\frac{\partial x_i}{\partial p_j} = \underbrace{\frac{\partial h_i}{\partial p_j}}_{\text{substitution}} - \underbrace{x_j \cdot \frac{\partial x_i}{\partial m}}_{\text{income effect}}$$

where h_i is the Hicksian (compensated) demand. The income effect equals the quantity consumed of the price-changing good, x_j , multiplied by the marginal propensity to consume good i out of income.

C.1 When Is the Income Effect Large or Small?

The income effect is large when:

1. **The good commands a large budget share.** Housing, for example. If rent rises 20%, you have meaningfully less real income—the wealth shock is large.
2. **The good is highly income-elastic (a luxury).** If your demand for the good rises steeply with income, then changes in real wealth translate strongly into demand changes.
3. **The price change itself is large.** A doubling of gasoline prices is a substantial real income shock for a commuter; a doubling of saffron prices is not.

The income effect is small when:

1. **The good is a small fraction of the budget**—toothpicks, salt, paper clips. The “real income” change from a price movement is negligible.
2. **The good is income-inelastic (a necessity with flat Engel curves).** Even if you get richer, you don’t consume much more of it.
3. **The good has close substitutes.** The compensated and uncompensated demands nearly coincide, so the Hicksian and Marshallian curves are close together.

References

- Auerbach, A. J. (1985). “The Theory of Excess Burden and Optimal Taxation.” *Handbook of Public Economics*, Vol. 1, Ch. 2.

- Chipman, J. S. and J. C. Moore (1980). “Compensating Variation, Consumer’s Surplus, and Welfare.” *American Economic Review*, 70(5), 933–949.
- Hausman, J. A. (1981). “Exact Consumer’s Surplus and Deadweight Loss.” *American Economic Review*, 71(4), 662–676.
- McKenzie, G. W. and I. F. Pearce (1976). “Exact Measures of Welfare and the Cost of Living.” *Review of Economic Studies*, 43(3), 465–468.
- Vives, X. (1987). “Small Income Effects: A Marshallian Theory of Consumer Surplus and Downward Sloping Demand.” *Review of Economic Studies*, 54(1), 87–103.
- Willig, R. D. (1976). “Consumer’s Surplus Without Apology.” *American Economic Review*, 66(4), 589–597.