Random Utility with Unobservable Alternatives

Haruki Kono, Kota Saito, Alec Sandroni*

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Abstract

The random utility model (RUM), a cornerstone in economics, is typically studied under the assumption that choice frequencies of all alternatives are observable. In practice, however, some alternatives have unobservable choice frequencies and are commonly aggregated into a single category called an outside option. We study RUM in such environments and derive a finite, nonredundant system of inequality constraints on observed choice frequencies that characterizes RU-rationalizability. We show that the conventional practice of aggregating unobserved alternatives can miss key information leading to incorrect conclusions such as that observed choices are rationalizable, even when no RUM is consistent with them.

Keywords: Random utility, axiom, network flow, polytope

*Kono: MIT, hkono@mit.edu. Saito: Caltech, saito@caltech.edu, Sandroni: Caltech, MIT, asandron@mit.edu. We appreciate the valuable discussions we had with Jean-Paul Doignon, Efe Ok, Faruk Gul, Wolfgang Pesendorfer, Pietro Ortoleva, Peter Klibanoff, Marciano Siniscalchi, and Takashi Ui. In particular, Jean-Paul Doignon as well as anonymous referees for EC 2023 read an earlier version of the manuscript and offered helpful comments. We also thank the editor, Pietro Ortoleva, and the anonymous referees of the American Economic Review for their comments and suggestions, which substantially improved the paper. We are grateful to William McCausland and Clintin Davis-Stober for generously providing the data used in this study. We also appreciate participants at presentations at Northwestern University, Princeton University, New York University, University of Tokyo, EC 2023, RUD 2023, and DTEA 2023. Kono acknowledges financial support from the Funai Overseas Scholarship and the Jerry A. Hausman Fellowship. Saito acknowledges financial support from the National Science Foundation through grant SES-1919263. Alec Sandroni acknowledges financial support from the Summer Undergraduate Research Fellowships (SURF) program in 2022, 2023, and 2024 at Caltech.

Consider a population of individuals choosing an alternative from various choice sets, where the analyst observes the choice frequencies of each alternative within each set. A foundational framework for interpreting such datasets is the random utility model (RUM), which posits a probability distribution over rankings of alternatives, with each ranking representing an individual's preferences. This model serves as a central tool in economics for linking observed stochastic choice behavior to underlying preference structures.

Falmagne (1978) axiomatized the RUM under the assumption that choice frequencies for all alternatives in all menus are observable. However, it is often the case that the choice frequencies for some alternatives are not observed. For example, consider a set of transportation methods consisting of bus, train, walking, and driving. While it may be possible to estimate the market share of public transportation (bus or train) based on the revenue of bus or train companies, it can be difficult to determine whether a person chooses to drive or walk unless a survey is conducted. As a result, the choice frequencies for walking and driving may not be available.

There are many other economically significant examples in which the choice frequencies of some alternatives are not observable. In such situations, empirical researchers often aggregate all unobservable alternatives into a single category and treat it as a generic *outside option*, even when they know which specific alternatives are available. We refer to this approach as *the outside option approach*. With this approach, the choice frequency of the outside option is recovered as one minus the sum of the observed frequencies.

The purpose of this paper is twofold. First, we investigate a necessary and sufficient condition for a RUM to rationalize the observed choice data when the choice frequencies of some alternatives are unobservable. Second, through this investigation, we demonstrate the limitations of the outside option approach. We show that by relying on the outside option approach, researchers may believe that a RUM applies even when it does not, and they may also overlook valuable information contained in the dataset. One key takeaway from these results is that empirical researchers should use the outside option approach with caution and make a deliberate effort to specify available alternatives in each choice set as much as possible. By doing so, researchers can exploit all the implications of the RUM. In Section I, we demonstrate these points by providing an example. The remainder of this subsection elaborates on these two objectives in turn.

We focus on a setup where the choice frequencies of some alternatives are consistently missing, while those for the other alternatives are observable. For such datasets, we derive a finite system of linear inequalities that provides a necessary and sufficient condition for the dataset to be rationalized by a RUM. Furthermore, the characterization we provide is nonredundant, meaning that none of the inequalities are implied by any others, and removing even one of them would render the condition insufficient.[1] As we explain later, the nonredundancy of the conditions helps us show the limitation of the outside option approach.

Our necessary and sufficient condition consists of two key components. The first is the classical nonnegativity of the Block-Marschak polynomials, a condition that appears in Falmagne's characterization. The second is a novel condition that consists of inequalities involving the summation and subtraction of the Block-Marschak polynomials. This novel condition highlights that the nonnegativity of the Block-Marschak polynomials alone is insufficient to rationalize a dataset. Instead, balances across the values of the Block-Marschak polynomials are crucial when dealing with incomplete datasets.

Our findings have significant implications for the widely used outside option approach. We show that the outside option approach disregards most inequalities of the second condition. In particular, our results show that even if the dataset is not rationalizable by any RUM, the outside option approach may erroneously conclude that agents' choices are rationalizable by a RUM.

To establish our results, we translate the problem into a network flow problem. This approach was originally developed by Fiorini (2004), who provided a shorter proof of the axiomatization of Falmagne (1978). Our methodological innovation is

[1] Falmagne (1978)'s characterization is also almost nonredundant. Suck (2002) and Fiorini (2004) show that omitting just a few inequalities from Falmagne (1978) results in a nonredundant characterization. The characterization by McFadden and Richter (1990) involves infinitely many inequalities and entails redundancy.

employing a feasibility theorem in network flow theory, which provides a necessary and sufficient condition for the existence of a desirable network flow. This novel tool offers clear insights even in cases where some alternatives are missing. Moreover, we demonstrate that our methodology not only facilitates characterization but also significantly enhances computational efficiency when testing our conditions in given datasets.

I Motivating Example

To demonstrate the importance of analyzing datasets without relying on the outside-option approach, we provide an example, in which the original dataset cannot be rationalized by any RUM, but under the outside option approach, the dataset appears to conform to a RUM. Moreover, in the example, with the outside option approach, we lose critical information about the desirability of alternatives.

Let $\{a, b, c, d\}$ be the set of alternatives. Suppose that we do not observe the choice frequencies of alternatives c and d. The left panel in Table 1 shows the observable choice frequencies $\rho(D, x)$ of alternative x from D. The right panel in Table 1 shows a reduced dataset $\hat{\rho}$ in which an outside option x_0 represents the set of all unobservable alternatives $\{c, d\}$. (See Definition 6 in Section IV.A for the formal definition of a reduced dataset.)

Note first that in Table 1, when there is only one unobservable alternative in a choice set, we can calculate its choice frequency by subtracting the sum of observable choice probabilities from one. For example, $\rho(\{a, c\}, c) = 1 - \rho(\{a, c\}, a) = 1/3$.

Note also that ρ cannot be rationalized by any RUM. One easy way to see this is that ρ violates *Monotonicity*, which requires that $\rho(D, x) \geq \rho(E, x)$ whenever $D \subseteq E$ and $x \in D \cap E$. RUM implies Monotonicity but the ρ in Table 1 violates this condition since $\rho(\{a, c\}, c) < \rho(\{a, b, c\}, c)$. Crucially however, this violation disappears in the reduced data $\hat{\rho}$ and in fact one can verify that $\hat{\rho}$ is rationalizable by a RUM, even though the original dataset is not.[2]

[2] The reduced dataset is rationalizable by a distribution μ such that $\mu(\succ_1) = 1/3$, $\mu(\succ_2) = 1/3$, $\mu(\succ_3) = 1/6$, and $\mu(\succ_4) = 1/6$, where $x_0 \succ_1 a \succ_1 b$, $x_0 \succ_2 b \succ_2 a$, $a \succ_3 b \succ_3 x_0$, and

ρ

$D \setminus x$	a	b	c	d
$\{a, b\}$	$1/2$	$1/2$	$-$	$-$
$\{a, c\}$	$2/3$	$-$	$1/3$	$-$
$\{a, d\}$	$2/3 + \varepsilon$	$-$	$-$	$1/3 - \varepsilon$
$\{b, c\}$	$-$	$1/2$	$1/2$	$-$
$\{b, d\}$	$-$	$1/2 + \varepsilon$	$-$	$1/2 - \varepsilon$
$\{c, d\}$	$-$	$-$	?	?
$\{a, b, c\}$	$1/3$	$1/6$	$1/2$	$-$
$\{a, b, d\}$	$1/3 + \varepsilon/2$	$1/6 + \varepsilon/2$	$-$	$1/2 - \varepsilon$
$\{a, c, d\}$	$1/3$	$-$	?	?
$\{b, c, d\}$	$-$	$1/3$	?	?
$\{a, b, c, d\}$	$1/6$	$1/6$	?	?

$\hat{\rho}$

$D \setminus x$	a	b	x_0
$\{a, b\}$	$1/2$	$1/2$	$-$
$\{a, x_0\}$	$1/3$	$-$	$2/3$
$\{b, x_0\}$	$-$	$1/3$	$2/3$
$\{a, b, x_0\}$	$1/6$	$1/6$	$2/3$

Table 1: The left panel shows the original choice probabilities $\rho(D, x)$ and the right panel shows the reduced choice probabilities $\hat{\rho}(D, x)$. The rows indicate choice set D, the columns indicate alternative x, and question marks indicate unobservable choice probabilities. We assume $0 < \varepsilon \leq 1/3$.

In the original dataset, $\rho(D, a) \geq \rho(D, b)$ for all D with $\{a, b\} \subseteq D$ and $\rho(D \cup a, a) \geq \rho(D \cup b, b)$ for any D such that $a \notin D$ and $b \notin D$, where these inequalities are strict when $D = \{c\}$ or $D = \{d\}$. Thus it can be inferred that a is strictly more desirable than b. However, in the reduced dataset, the desirability of a and b is indistinguishable. This illustrates how the estimated desirability of a and b may be biased when using the outside-option approach.

Another consequence of the outside option approach is that in the reduced dataset c and d are indistinguishable. In the original dataset, however, we can compare choice sets with different unobservable alternatives to learn about their relative desirability. In particular, we can learn that c is more desirable than d. For example, we have that $\rho(D \cup c, c) > \rho(D \cup d, d)$ for all nonempty D such that $c \notin D$ and $d \notin D$.

Furthermore, under additional structure, we can draw a stronger conclusion about the choice frequencies of c and d even when both are in the choice

$b \succ_4 a \succ_4 x_0$.

set. For example, suppose we only observe the choice sets $\{b, c, d\}$, $\{b, c\}$, and $\{b, d\}$. Assuming Monotonicity, we have $\rho(\{b, c, d\}, d) \leq \rho(\{b, d\}, d) = 1/2 - \varepsilon$, which gives us an upper bound for $\rho(\{b, c, d\}, d)$. On the other hand, a lower bound for $\rho(\{b, c, d\}, c)$ is obtained as follows: $\rho(\{b, c, d\}, c) = 1 - \rho(\{b, c, d\}, b) - \rho(\{b, c, d\}, d) \geq 1 - 1/3 - (1/2 - \varepsilon) = 1/6 + \varepsilon$. Therefore when $\varepsilon > 1/6$, we have that $\rho(\{b, c, d\}, d) \leq 1/2 - \varepsilon < 1/3 < 1/6 + \varepsilon \leq \rho(\{b, c, d\}, c)$. We conclude that, under Monotonicity (and therefore RUM) the probability that c is chosen from $\{b, c, d\}$ is higher than the probability that d is chosen from the same menu. All of these conclusions suggest that c is more desirable than d, information that is lost in the reduced dataset.

In this example, there are only two unobservable alternatives, so we can directly infer the choice frequency of each unobservable alternative when the choice set contains only one unobservable alternative. However, even when there are more than two unobservable alternatives in a choice set, we can still learn about the sum of their choice frequencies, which enables us to compare the desirability of different unobservable alternatives. In this sense, our conclusion—that identifying which alternatives are available to the agent in each menu allows us to infer more about the agent's preferences—does not hinge on the special case with only two unobservable alternatives in the dataset. In fact, we formalize the results in Section V.

These observations underscore the limitations of the outside-option approach. In this paper, we formalize those limits by first characterizing the full implications of RUM when some choice frequencies are unobservable (Theorem 1) and by identifying which implications are lost under the outside-option approach (Proposition 1). Moreover, in Section V, we formalize our methods to obtain bounds for unobserved choice frequencies. In particular, we provide an efficient algorithm that exploits the problem's network-flow structure to compute tight bounds, offering a practical tool for further analysis. Using real datasets, we show that our bounds are substantially tighter than the naive bounds derived from the outside-option approach and that they recover information about the relative desirability of unobservable alternatives that the outside-option approach entirely discards (see

Remark 6 and Proposition 2).

II Related Literature

It is well known that obtaining a nonredundant characterization of the RUM with incomplete datasets in general is a challenging problem. See Martí and Reinelt (2011) for a survey. A more recent paper by Sprumont (2022) also highlights the difficulty of this problem. For example, when choice frequencies are observed only on binary choice sets, it has been unknown how to obtain a nonredundant characterization of the RUM since the 1980s; only for the case where the number of alternatives is less than eight, the nonredundant characterizations have been obtained.[3]

McFadden and Richter (1990) proposed a characterization of the RUM for incomplete datasets. However, their characterization involves infinitely many inequalities and entails redundancy, while ours consists of finite inequalities and contains no redundant ones.[4] Moreover, a well-known version of McFadden and Richter's axiom is not sufficient to characterize RUM in the presence of unobservable alternatives, although the original version of their axiom may be adapted to our setup. See Section G.3 of the supplemental appendix for details.

Other than the papers mentioned so far, only a few studies have investigated the characterization of the RUM with incomplete data. McFadden (2005) considers a nested structure of choice sets: if choice frequencies are observable in a menu D, then choice frequencies are observable in any larger set E (i.e., $E \supseteq D$). In this paper, we also adopt this restriction of available choice sets. Suck (2016) addresses the truncated complete choice environment, in which only choice sets with at least $k \geq 2$ alternatives are observable. Nevertheless, to the best of our knowledge, our

[3]See Reinelt (1993). This contrasts with logit model: logit models can be axiomatized with binary choice sets. See Luce (1959), Cerreia-Vioglio et al. (2023), and Cerreia-Vioglio et al. (2021) for the axiomatization of the logit models. Recently, Petri (2023) has obtained an axiomatization of a special case of the RUM, *single-crossing RUMs* introduced by Apesteguia, Ballester and Lu (2017) in a binary choice setup.

[4]More recently, Turansick (2023) provides an alternative axiomatization using the network flow approach.

setup, in which the choice frequencies of some alternatives are missing, is novel in the literature. Moreover, these results are special cases of our theorem—cases in which there are no unobservable alternatives.

As mentioned, we use the network-flow theory to prove our results. Since the publication of Fiorini (2004), some more recent papers have used the network-flow theory to investigate different topics on RUM. Turansick (2022) and Chambers and Turansick (2025) study the identification of RUMs. Chambers, Masatlioglu and Turansick (2024) provide a new model of random utility with more than one agent. Doignon and Saito (2023) characterize the adjacency of vertices and facets of a *multiple-choice polytope* (i.e., the set of RUMs). None of these papers study incomplete datasets.

Finally, we highlight several studies that develop empirical methods for testing RUMs. Kitamura and Stoye (2018) and Dean, Ravindran and Stoye (2022) propose nonparametric statistical tests for RUMs. Their methods, however, are computationally intensive because they rely on a vertex representation of the RUM polytope; the number of vertices can be enormous.[5] By contrast, our characterization exploits the specific structure of data incompleteness and the associated network-flow formulation, yielding more computationally efficient testing methods based on a nonredundant half-space representation rather than an explicit enumeration of vertices. See Section G.2 of the supplemental appendix for further discussion.

III Model

Let X be a finite set of alternatives. Let $X^* \subseteq X$ be the set of *unobservable alternatives*. We assume that the choice frequencies of the elements of X^* are not observable (even if a choice set includes the alternatives). Let $\tilde{X} \equiv X \setminus X^*$ be the set of *observable alternatives*. We assume $|\tilde{X}| > 1$.

Let $\mathcal{D} \subseteq 2^X \setminus \{\emptyset\}$ be the set of choice sets. We assume that $\tilde{X} \in \mathcal{D}$.[6] Unlike

[5] More recently, Koida and Shirai (2024) characterize RUM in terms of hyperplanes, but they focus on datasets generated from budget sets.

[6] We assume $|\tilde{X}| > 1$ and $\tilde{X} \in \mathcal{D}$ so that the standard model without unobservable alternatives is nested as a special case. In particular, when there are no unobservable alternatives (i.e.,

Falmagne (1978) and Fiorini (2004), we do not assume $\mathcal{D} = 2^X \setminus \{\emptyset\}$.

Note that $(\mathcal{D}, \subseteq)$ is a partially ordered set, where $\subseteq$ is the set inclusion. Like McFadden (2005), we assume that $\mathcal{D}$ is an *upper set* (i.e., $\mathcal{D}$ satisfies the following: $D \in \mathcal{D},\ X \supseteq E \supseteq D \Longrightarrow E \in \mathcal{D}$). To make our notation simple, let $\mathcal{M} \equiv \{(D, x) \in \mathcal{D} \times \tilde{X} \mid x \in D\}$.

Note also that for any (D, x), the choice frequency of x from D is observable if and only if $(D, x) \in \mathcal{M}$ (i.e., $x \in \tilde{X}$ and $D \in \mathcal{D}$).

Definition 1. *A nonnegative vector $\rho \in \mathbf{R}_+^{\mathcal{M}}$ is called an incomplete dataset if it satisfies the following conditions: for any $D \in \mathcal{D}$,*

(i) if $D \subseteq \tilde{X}$, then $\sum_{x \in D} \rho(D, x) = 1$; and

(ii) if $D \not\subseteq \tilde{X}$, then $\sum_{x \in D \cap \tilde{X}} \rho(D, x) \leq 1$.

When the context is clear, we will simply call ρ a *dataset* instead of an *incomplete dataset*. If ρ is an incomplete dataset, then ρ is not defined on $(D, x) \notin \mathcal{M}$. This does not mean that we cannot know anything about the choice frequencies of elements in X^*. We can compute the total choice frequency of all unobservable alternatives in D as $1 - \sum_{x \in D \cap \tilde{X}} \rho(D, x)$.

Definition 2. *A nonnegative vector $\rho^* \in \mathbf{R}_+^{\{(D,x) | x \in D \in 2^X\}}$ is called a complete dataset if, for any nonempty $D \subseteq X$, $\sum_{x \in D} \rho^*(D, x) = 1$.*

A Examples

Example 1 (Transportation): An analyst is often able to estimate the market share of public transportation methods (i.e., bus or train) based on the revenues of bus or train companies. However, it is sometimes difficult for the analyst to know separately the percentages of people who drive or walk. In this case, $X = \{\text{walk, drive, bus, train}\}$, $\tilde{X} = \{\text{bus, train}\}$ and $X^* = \{\text{walk, drive}\}$. An example of the choice sets is $\mathcal{D} = \{\{b, t\}, \{d, b, t\}, \{w, b, t\}, \{w, d, b\}, \{w, d, t\}, \{w, d, b, t\}\}$,

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$X = \tilde{X}$), the first assumption reduces to the nontriviality condition $|X| > 1$ assumed in Fiorini (2004); the second assumption reduces to $X \in \mathcal{D}$, which is implied by the standard full domain assumption (i.e., $\mathcal{D} = 2^X \setminus \{\emptyset\}$).

where w, d, b, and t stand for *walk*, *drive,* *bus*, and *train*, respectively. This set $\mathcal{D}$ can be obtained from the assumption that depending on the location of homes, some transportation methods are not available.[7]

Example 2 (Market Shares of Private Companies): One definition of market share is a company's sales divided by the market's total sales. The market's total sales can be estimated by consumer surveys. However, private companies occasionally do not disclose their financial information, including their total sales; thus the market shares of private companies are sometimes unobservable. For example, suppose that there are four companies (i.e., $X = \{a, b, c, d\}$). If companies c and d are private companies, then we do not know their sales. Other companies $\{a, b\}$ are public and the information from these companies is disclosed. That is, $\tilde{X} = \{a, b\}$ and $X^* = \{c, d\}$. In addition, the availability of products may vary across stores, which would generate variation of choice sets (i.e., $\mathcal{D}$).

Example 3 (School Choice for Private Schools): Applicants submit their choices among public schools so the government knows the percentage of students choosing each public school. However, it might not have access to information on how many students choose each private school. For example, suppose that there are four schools (i.e., $X = \{a, b, c, d\}$). Among them, c and d are private schools for which we do not know the choice frequencies. Thus, $\tilde{X} = \{a, b\}$ and $X^* = \{c, d\}$. The availability of schools may depend on the location of homes, which would generate variation of choice sets (i.e., $\mathcal{D}$).

B Random Utility Rationalization

Let $\mathcal{L}$ be the set of linear orders on X, i.e., binary relations that are irreflexive, asymmetric, transitive, and weakly complete (i.e., for any distinct elements $x, y \in X$, either $x \succ y$ or $y \succ x$). We interpret each linear order as representing an agent's strict preference ordering in the population.

[7]Assuming that the analyst believes that the distribution of preferences is independent of the location of homes, it would make sense to find a single distribution over preferences that describes the choice frequencies across $\mathcal{D}$.

Definition 3. *An incomplete dataset ρ is random-utility (RU) rationalizable if there exists $\mu \in \Delta(\mathcal{L})$ such that, for any $(D,x) \in \mathcal{M}$, $\rho(D,x) = \mu(\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x)$. We then say that μ rationalizes ρ.*[8]

Definition 4. *Let $p \in \mathbf{R}^{\{(D,x) \mid x \in D \in 2^X\}}$. For any (D,x) such that $x \in D \subseteq X$, define $K(p,D,x) \equiv \sum_{E:E\supseteq D} (-1)^{|E \setminus D|} p(E,x)$. $K(p,D,x)$ is called a Block-Marschak (BM) polynomial.*[9] *For an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$, define BM polynomials analogously whenever they are well-defined.*

Note that, given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$, the BM polynomial $K(\rho,D,x)$ can be calculated if and only if $(D,x) \in \mathcal{M}$ (i.e., $x \in \tilde{X}$ and $D \in \mathcal{D}$). The next remark provides an interpretation of a BM polynomial given a complete dataset ρ^*:

Remark 1. *Assuming that a complete dataset ρ^* is RU-rationalizable, we can provide an interpretation of a BM polynomial. Suppose that there exists $\mu \in \Delta(\mathcal{L})$ such that, for any (D,x) such that $x \in D \subseteq X$, $\rho^*(D,x) = \mu(\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x)$. By the Möbius inversion formula, it follows that*

$$K(\rho^*, D, x) = \mu(\succ \in \mathcal{L} \mid D^c \succ x \succ D \setminus x), \quad (1)$$

where $D^c \succ x$ means that $y \succ x$ for all $y \in D^c$ and $x \succ D \setminus x$ means that $x \succ y$ for all $y \in D \setminus x$.[10] *Thus, $K(\rho^*, D, x)$ can be interpreted as the measure of agents whose preference satisfies $D^c \succ x \succ D \setminus x$.*

IV Main Results

To characterize the RU-rationalizability of incomplete data, we define the following collection of choice sets:

[8] To simplify notation, we write $D \setminus x$ and $D \cup x$ as shorthand for $D \setminus \{x\}$ and $D \cup \{x\}$, respectively, whenever the meaning is clear.

[9] As we will explain below, the BM polynomial is a crucial concept to characterize RUMs. The BM polynomial appears in other contexts. For example, Brady and Rehbeck (2016) observe that one of their axioms is equivalent to a multiplicative version of the BM polynomial.

[10] To see this notice that $\mu(\succ \in \mathcal{L} \mid x \succ D \setminus x) = \mu(\bigcup_{E \supseteq D} \{\succ \in \mathcal{L} \mid E^c \succ x \succ E \setminus x\}) = \sum_{E \supseteq D} \mu(\succ \in \mathcal{L} \mid E^c \succ x \succ E \setminus x)$. Thus, we have $\rho^*(D,x) = \sum_{E \supseteq D} \mu(\succ \in \mathcal{L} \mid E^c \succ x \succ E \setminus x)$. By applying the Möbius inversion to this equation, we obtain $\mu(\succ \in \mathcal{L} \mid D^c \succ x \succ D \setminus x) = K(\rho^*, D, x)$.

Definition 5. *A nonempty collection $\mathcal{C}$ of subsets of X is called a test collection if there exists a set $A \subseteq \tilde{X}$ of observable alternatives and a nonempty upper set $\mathcal{E} \subseteq 2^{X^*}$ of unobservable alternatives such that $\mathcal{C} = \{A \cup E \mid E \in \mathcal{E}\}$.*[11] *Moreover, the test collection is said to be essential if $\emptyset \neq A \neq \tilde{X}$ and $\mathcal{E} \neq 2^{X^*}$.*

That is, a test collection is a set of menus that contain the same subset of observable alternatives and is closed under the addition of unobservable alternatives. The following is our main theorem.

Theorem 1.

(a) *An incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$ is RU-rationalizable if and only if the following two conditions hold:*

- *(i) for any $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$, the BM polynomial $K(\rho, D, x)$ is nonnegative; and*
- *(ii) for any essential test collection $\mathcal{C} \subseteq \mathcal{D}$,*

$$(2) \qquad \sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D\cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F\cup y, y) \geq 0.$$

(b) *The inequality conditions in (i) and (ii) are independent: for any inequality condition in (i) or (ii), there exists an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$ that violates the inequality but satisfies all the other conditions in (i) and (ii). (Note that by statement (a), such ρ is not RU-rationalizable.)*

Recall that for any (D, x), the BM polynomial $K(\rho, D, x)$ is computable based on the observable data if and only if $(D, x) \in \mathcal{M}$. Thus, condition (i) is testable. Also, when $\mathcal{C} \subseteq \mathcal{D}$ is a test collection, $D \in \mathcal{C}$ and $D \cup x \notin \mathcal{C}$ imply that $x \in \tilde{X}$ and the upper set property of $\mathcal{D}$ implies that $D \cup x \in \mathcal{D}$.[12] Thus, the first term as well

[11] Recall the definition of an upper set: if $D \in \mathcal{E}, D \subseteq E \subseteq X^* \implies E \in \mathcal{E}$. In the example in which $X^* = \{d, e\}$, all upper sets in 2^{X^*} are $\emptyset$, $\{\{d, e\}\}$, $\{\{d, e\}, \{d\}\}$, $\{\{d, e\}, \{e\}\}$, $\{\{d, e\}, \{d\}, \{e\}\}$, and $\{\{d, e\}, \{d\}, \{e\}, \emptyset\}$.

[12] Since $\mathcal{C}$ is a test collection, $\mathcal{C} = \{A \cup E \mid E \in \mathcal{E}\}$ for some $A \subseteq \tilde{X}$ and an upper set $\mathcal{E} \subseteq 2^{X^*}$. If $x \in X^*$, then $D \in \mathcal{C}$ implies $D \cup x \in \mathcal{C}$ by the definition of test collections (especially by the fact that $\mathcal{E}$ is an upper set).

as the second term in condition (ii) can be calculated based on the available data; so condition (ii) is also testable.

When all alternatives are observable (i.e., $\tilde{X} = X$, equivalently $X^* = \emptyset$), our theorem specializes to the classical result of Falmagne (1978): a complete dataset is RU-rationalizable if and only if all Block–Marschak (BM) polynomials are nonnegative. Accordingly, condition (i) of our theorem requires nonnegativity of the nontrivial BM polynomials that are computable from the incomplete data.

Novel conditions appear in (ii), which means that the nonnegativity of the BM polynomials is insufficient for the dataset to be RU-rationalizable because balances across the values of BM polynomials are essential for RU-rationalizability when the dataset is incomplete. For instance, a single BM value that is too large can trigger a violation if it appears with a negative coefficient. In Remark 2 and Subsection IV.B, we provide a further explanation of condition (ii).

Statement (b) of the theorem says that each inequality condition in (i) or (ii) cannot be implied by the other inequality conditions; thus each can be interpreted individually. As we explain later in Proposition 1, this feature is crucial to understand which implications of RUM may be lost in the outside option approach.

Remark 2. *Suppose that an incomplete dataset ρ is RU-rationalizable by μ. To understand the interpretation of condition (ii) in Theorem 1, fix a test collection $\mathcal{C}$ and assume that $\mathcal{C}$ is a singleton set containing a menu D. Since $\mathcal{C}$ is a test collection, we have $X^* \subseteq D$. Then, the left-hand side of (2) simplifies to*

$$\sum_{x^* \in D \cap X^*} \mu(\succ \in \mathcal{L} \mid D^c \succ x^* \succ D \setminus x^*). \tag{3}$$

Thus the interpretation of condition (ii) for this case is the nonnegativity of the measure of agents whose preference satisfies $D^c \succ x^* \succ D \setminus x^*$ for some $x^* \in X^*$.

Note that there is no testable implication of the nonnegativity of $\mu(\succ \in \mathcal{L} \mid D^c \succ x^* \succ D \setminus x^*)$ for each $x^* \in X^*$, because $K(\rho, D, x^*)$ cannot be computed, unlike in the complete-data case (Remark 1). However, we can still compute $\sum_{x^* \in D \cap X^*} K(\rho, D, x^*)$, which coincides with the left-hand side of (2) and admits the interpretation in terms of μ given in (3). For the general form of condition (2) *and the proof, see Section G.1 of the supplemental appendix.*

A Implication for the outside option approach

In the outside option approach, we represent the set X^* of all unobservable alternatives as a single alternative x_0. Thus, the set of all alternatives in the outside option approach is defined as $\hat{X} = \tilde{X} \cup x_0$, where $\tilde{X}$ is the set of observable alternatives. Consequently, we consider the *reduced* choice sets, denoted by $\hat{\mathcal{D}}$, which aggregates all elements of X^* into the outside option x_0. We ignore the data on choice sets that contain only some (but not all) elements of X^*. Formally, $\hat{\mathcal{D}} \equiv \{D \in \mathcal{D} \mid D \cap X^* = \emptyset\} \cup \{(D \setminus X^*) \cup x_0 \mid D \in \mathcal{D}, D \supseteq X^*\} \subseteq 2^{\hat{X}}$.

Definition 6. *Let $\hat{\rho} \in \mathbf{R}_+^{\{(D,x) | x \in D \in \hat{\mathcal{D}}\}}$ be the reduced dataset on the reduced choice sets defined as follows. For all $\hat{D} \in \hat{\mathcal{D}}$, define $D = (\hat{D} \cap \tilde{X}) \cup X^*$ if $x_0 \in \hat{D}$ and $D = \hat{D}$ otherwise. We define the choice frequencies of observable alternatives to remain the same, i.e., $\hat{\rho}(\hat{D}, x) = \rho(D, x)$ for all $x \in D \cap \tilde{X}$, where $\rho \in \mathbf{R}_+^{\mathcal{M}}$ is the original (non-reduced) incomplete dataset. Then, the choice frequency of the outside option can be defined as $\hat{\rho}(\hat{D}, x_0) = 1 - \sum_{x \in D \cap \tilde{X}} \rho(D, x)$.*

See Table 1 in Subsection I for the example of $\hat{\rho}$. We say that the reduced dataset $\hat{\rho} \in \mathbf{R}_+^{\{(\hat{D},x) | x \in \hat{D} \in \hat{\mathcal{D}}\}}$ is *RU-rationalizable* if there exists a probability distribution $\hat{\mu}$ on the set $\hat{\mathcal{L}}$ of linear orders on $\hat{X}$ such that for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$, $\hat{\rho}(\hat{D}, x) = \hat{\mu}(\hat{\succ} \in \hat{\mathcal{L}} \mid x \mathrel{\hat{\succ}} y \text{ for all } y \in \hat{D} \setminus x)$.

Proposition 1. *Let $\rho \in \mathbf{R}_+^{\mathcal{M}}$ be an incomplete dataset. Suppose that ρ satisfies*

- *condition (i) of Theorem 1 and*
- *condition (ii) of Theorem 1 for any singleton essential test collection $\mathcal{C} \subseteq \mathcal{D}$.*

Then the reduced dataset $\hat{\rho} \in \mathbf{R}_+^{\{(\hat{D},x) | x \in \hat{D} \in \hat{\mathcal{D}}\}}$ is RU-rationalizable.

For the RU-rationalizability of the reduced dataset $\hat{\rho}$, Proposition 1 requires condition (ii) only for *singleton* essential test collections (in addition to condition (i)), whereas for the RU-rationalizability of the original dataset ρ, Theorem 1 imposes condition (ii) for *all* essential test collections (in addition to condition (i)).

To see the implication of the gap between Proposition 1 and Theorem 1 more clearly, suppose the original incomplete dataset ρ violates condition (ii) of Theorem 1 for some *non-singleton* essential test collection, while satisfying all other conditions of Theorem 1. Note that such ρ exists because of the independence of each inequality as stated in statement (b) of Theorem 1. Statement (a) of Theorem 1 implies that ρ is not RU-rationalizable. Nevertheless, Proposition 1 implies that the reduced dataset $\hat{\rho}$ is RU-rationalizable, thus researchers may erroneously conclude that the true data-generating process follows the RUM. In this way, the reduced dataset $\hat{\rho}$ loses critical implications of the original dataset ρ. In particular, Proposition 1 shows that the outside option approach discards all but the most basic inequalities in condition (ii).

Remark 3. *To illustrate the implication of Proposition 1, consider the variant of the incomplete dataset ρ in Table 1 obtained by setting $\varepsilon = 0$.*

- *One can observe that ρ violates condition (ii) with $\mathcal{C} = \{\{a, c, d\}, \{a, c\}\}$:*

$$\sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D \cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F \cup y, y) = -\frac{1}{6}$$

- *Therefore, by Theorem 1, the original dataset ρ is not RU-rationalizable. However, since ρ satisfies condition (i) and condition (ii) when $\mathcal{C}$ is a singleton, Proposition 1 implies that the reduced dataset $\hat{\rho}$ is RU-rationalizable, despite the fact that the true dataset ρ is not.*

Remark 4. *Our definition assumes that the composition of the outside option is constant across choice sets—that is, x_0 always represents the same set X^* of unobservable alternatives. Although this corresponds to the idealized setup often assumed in the empirical literature, it may not hold in real datasets. In practice, the interpretation of x_0 can vary across choice sets, and this variation is typically unobservable to the analyst. Liao, Saito and Sandroni (2025) examine such cases by taking the reduced dataset as the primitive of the model. That paper shows that*

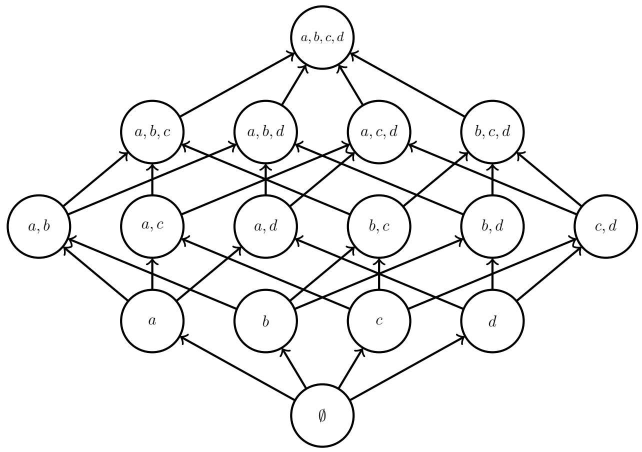

Figure 1: Network flow for the case in which $X = \{a, b, c, d\} = \tilde{X}$ and $X^* = \emptyset$ and $\mathcal{D} = 2^X \setminus \{\emptyset\}$.

the implications of RUM are further weakened. See Section E in the appendix for more details.

B Intuition of the proof

In this subsection, we outline the proof of statement (a) of Theorem 1. All formal proofs are in the appendix.

Before providing the sketch of our proof, we provide an overview of the network flow approach used by Fiorini (2004) for a complete dataset ρ^*. Recall that a *network* is a pair of a node set $\mathcal{N}$ and a set of directed arcs (i.e., edges) $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$. Two nodes s (source) and t (terminal) play special roles as explained below. We set $\mathcal{N} = 2^X$, $\mathcal{A} = \{(D, D \cup x) \mid D \subseteq X, x \notin D\}$, $s = \emptyset$, and $t = X$. See Figure 1 for an example. In the setup, each $\emptyset - X$ directed path corresponds to a unique ranking $\succ$. For instance, the directed path $\emptyset - \{a\} - \{a, b\} - \{a, b, c\} - X$ corresponds to the ranking: $d \succ c \succ b \succ a$.

We now construct a vector (i.e., *flow*) $r \in \mathbf{R}_+^{\{(D \setminus x, D) | x \in D \in 2^X\}}$ on the network assuming that a complete dataset ρ^* is RU-rationalizable by some $\mu \in \Delta(\mathcal{L})$, or

(4) $\quad \rho^*(D, x) = \mu(\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x) \text{ for all } (D, x) \text{ s.t. } x \in D \subseteq X.$

To each arc of the network, we assign the sum of the values of $\mu(\succ)$ over linear orders $\succ$ such that the directed path corresponding to $\succ$ goes through the arc. Given the construction, the value $r(D \setminus x, D)$ at an arc $(D \setminus x, D)$ is $\mu(\succ \in \mathcal{L} \mid D^c \succ x \succ D \setminus x)$.[13] By the Möbius inversion formula, as explained in Remark 1, this value equals $K(\rho^*, D, x)$. Note the constructed flow r satisfies all the following constraints:

$$(5) \quad \sum_{x \in X} r(X \setminus x, X) = 1,$$

$$(6) \quad \sum_{x \in D} r(D \setminus x, D) = \sum_{y \notin D} r(D, D \cup y) \text{ for any } D \in 2^X \text{ s.t. } 1 \leq |D| \leq |X| - 1,$$

$$(7) \quad r(D \setminus x, D) = K(\rho^*, D, x) \text{ for all } (D, x) \text{ such that } x \in D \in 2^X.$$

Condition (5) means that the sum of *inflows* to X (i.e. flows going into X) must be one; condition (6) means that for each node D, the sum of inflows to the node D equals the sum of *outflows* from D (i.e. flows coming out from D); finally condition (7) means that the value of flow at an arc equals the corresponding value of the BM polynomial.

The above observation shows that the three conditions are necessary for ρ^* to be RU-rationalizable. Fiorini (2004) proved that the conditions are also sufficient:

Lemma 1. *Given a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x) \mid x \in D \in 2^X\}}$, there exists $\mu \in \Delta(\mathcal{L})$ satisfying (4) if and only if there exists $r \in \mathbf{R}_+^{\{(D \setminus x, D) \mid x \in D \in 2^X\}}$ satisfying (5), (6) and (7).*

It can be shown from the definition of the BM polynomial that (5) and (6) are satisfied automatically under the assumption of (7). Hence:

Theorem (Falmagne (1978))**.** *Given a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x) \mid x \in D \in 2^X\}}$, there exists $\mu \in \Delta(\mathcal{L})$ satisfying (4) if and only if $K(\rho^*, D, x) \geq 0$ for all (D, x) such that $x \in D \in 2^X$.*

[13] To see this, note that $\succ$ passes through the arc $(D \setminus x, D)$ if and only if x is ranked just above the elements of $D \setminus x$. That is, $\succ$ passes through the arc $(D \setminus x, D)$ if and only if $D^c \succ x \succ D \setminus x$. For example, the path corresponding to $\succ$ passes through the arc $(\emptyset, \{a\})$ if and only if a is the worst element under $\succ$. Therefore the value assigned to the arc is equal to a measure over rankings whose worst element is a.

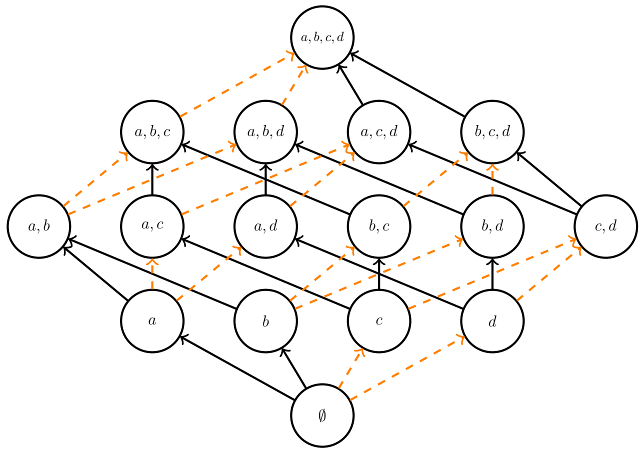

Figure 2: Network flow for the case in which $X = \{a, b, c, d\}$, $\tilde{X} = \{a, b\}$, and $X^* = \{c, d\}$ and $\mathcal{D} = 2^X \setminus \{\emptyset\}$. The solid arrows correspond to observable arcs; the dotted arrows correspond to unobservable arcs.

Now we will consider the case in which the dataset is incomplete. Remember in this case that $K(\rho, D, x)$ can be calculated if and only if $(D, x) \in \mathcal{M}$. We say an arc $(D \setminus x, D)$ is *observable* if $(D, x) \in \mathcal{M}$ (and hence we can calculate the value $K(\rho, D, x)$ of a flow at the arc); $(D \setminus x, D)$ is *unobservable* if $(D, x) \notin \mathcal{M}$ (and hence we *cannot* calculate the value $K(\rho, D, x)$ of a flow at the arc). See Figure 2, for illustration.

We extend the result of Fiorini (2004) to incorporate the case of incomplete datasets as follows:

Lemma 2. *Given an incomplete dataset* $\rho \in \mathbf{R}_+^{\mathcal{M}}$,

(P1) *there exists* $\mu \in \Delta(\mathcal{L})$ *such that*

(8) $$\rho(D, x) = \mu(\succ \in \mathcal{L} \mid x \succ y \textit{ for all } y \in D \setminus x) \textit{ for any } (D, x) \in \mathcal{M}$$

if and only if

(P2) *there exists* $r \in \mathbf{R}_+^{\{(D \setminus x, D) | x \in D \in 2^X\}}$ *satisfying (5), (6), and*

(9) $$r(D \setminus x, D) = K(\rho, D, x) \textit{ for all } (D, x) \in \mathcal{M}.$$

Note that unlike the case of complete data, Fiorini's approach (i.e., map-

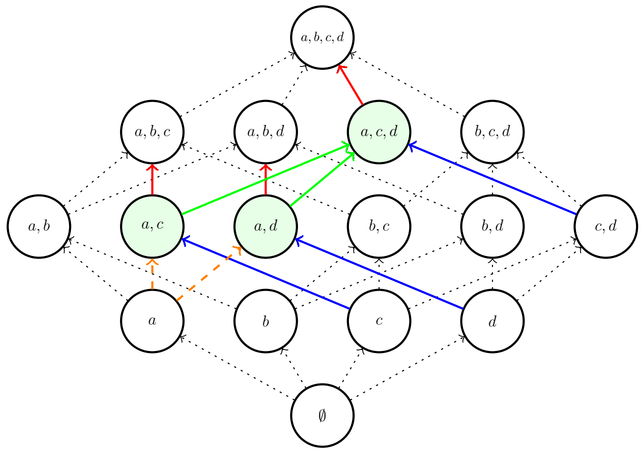

Figure 3: Outflows from $\mathcal{C} \equiv \{\{a,c\},\{a,d\},\{a,c,d\}\}$ and inflows to $\mathcal{C}$. Red flows are observable outflows from $\mathcal{C}$; yellow flows are unobservable inflows to $\mathcal{C}$; blue flows are observable inflows to $\mathcal{C}$. Note that green flows are flows contained in $\mathcal{C}$ and are not relevant to the net observable outflow from $\mathcal{C}$.

ping the RU-rationalizability problem into a network flow) does not provide direct testable conditions without existential quantifiers.

In the following, we translate $(P2)$ to conditions without existential quantifiers. To provide an intuition on how to do so, consider Figure 3 in which $X = \{a,b,c,d\}$, $\tilde{X} = \{a,b\}$, $X^* = \{c,d\}$, and $\mathcal{D} = 2^X \setminus \{\emptyset\}$. Let $\mathcal{C} = \{\{a,c\},\{a,d\},\{a,c,d\}\}$. In the figure, red flows are observable outflows from $\mathcal{C}$; yellow flows are unobservable inflows to $\mathcal{C}$; blue flows are observable inflows to $\mathcal{C}$. Note that there are no unobservable outflows by the choice of $\mathcal{C}$.

By the equality between inflows and outflows with respect to $\mathcal{C}$, we have that the sum of the red outflows equals the sum of the yellow inflows and the blue inflows. Although each yellow inflow is unobservable, we can calculate the sum of the yellow inflows as the sum of the red outflows minus the sum of the blue inflows. If ρ is RU-rationalizable then each yellow inflow is nonnegative, and hence *the sum of the red outflows minus the sum of the blue inflows must be nonnegative.* This is an implication that is *directly testable* based on the observable dataset since the blue inflows and the red outflows are observable. In fact, they can be calculated as follows: (Red outflows)$= \sum_{(D,x):D\in\mathcal{C}, D\cup x\notin\mathcal{C}} K(\rho, D\cup x, x)$; (Blue inflows)= $\sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F\cup y, y)$. Thus, the testable implication can be written as follows

$$(10) \qquad \sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D\cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F\cup y, y) \geq 0,$$

which means that *the net observable outflows from $\mathcal{C}$ must be nonnegative; and this is exactly one of the inequalities appearing in condition (ii) in Theorem 1.*

V Bounds for Unobservable Choice Probabilities

In this section, we derive bounds on unobservable choice probabilities, which are often needed for policy analysis. For example, in the transportation setting (Example 1 in Section III), commuting choices are observed only for public transit users: $X = \{\text{bus}, \text{train}, \text{walk}, \text{drive}\}$, $\tilde{X} = \{\text{bus}, \text{train}\}$, and $X^* = \{\text{walk}, \text{drive}\}$. If the government considers a gasoline tax to promote public transit, it is crucial to know the share of commuters who drive.

The most naive approach is merely to bound the fraction from below by zero and from above by the percentage of people who did *not* use public transportation. In Remark 5, we will observe that this naive approach corresponds to the outside option approach.

Remark 5. *Let $\rho \in \mathbf{R}_+^{\mathcal{M}}$ be a given incomplete dataset. Let $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in\hat{\mathcal{D}}\}}$ be the reduced dataset in the outside option approach defined in Definition 6. Choose any (D, x^*) such that $D \in \mathcal{D}$, $X^* \subseteq D$, and $x^* \in X^*$. If $|X^*| > 1$, then the bounds for the unobservable choice frequency of x^* from D derived from the restriction that $\hat{\rho}$ is RU-rationalizable are*

$$(11) \qquad \Big[0, 1 - \sum_{a\in D\cap\tilde{X}} \rho(D, a)\Big].$$

To see this, notice that every menu in the reduced dataset $\hat{\mathcal{D}}$ either contains all or none of the unobservable alternatives. Therefore, there is no way to distinguish any unobservable alternative from another. Thus, with the outside option *approach, RU-rationalizability does not have any implications beyond the fact that the probabilities must sum to* 1.

A Bounds derived from Monotonicity

In this section and the next, we show how the naive bounds can be greatly tightened by exploiting implications of RUM. Here we focus on the simplest one–Monotonicity: for all $D, E \in \mathcal{D}$ and $x \in D \cap E$, if $D \subseteq E$ then $\rho(D, x) \geq \rho(E, x)$. Monotonicity simply means that the choice probability of an alternative will not increase if a new alternative is added to a choice set.

Fix $D \in \mathcal{D}$ that contains some unobservable alternative $x^* \in X^*$. We will obtain the bound for the choice frequency of x^* from choice set D. The basic idea is that by comparing the choice probabilities on the choice sets D and $D \setminus y^*$ for each unobservable alternative $y^* \in D \cap X^*$, we can learn about the relative desirability of each unobservable alternative $y^* \in D \cap X^*$.

Remark 6. *Let ρ^* be a complete dataset that coincides with ρ on $\mathcal{M}$ and satisfies Monotonicity. Define $F \equiv \{y^* \in D \cap X^* \mid D \setminus y^* \in \mathcal{D}\}$. For any (D, x^*) such that $x^* \in D \cap X^*$ and $D \in \mathcal{D}$, the bounds for the unobservable choice frequency $\rho^*(D, x^*)$ become*

$$\left[L(x^*), 1 - \sum_{a \in D \cap \tilde{X}} \rho(D, a) - \sum_{y^* \in (D \cap X^*) \setminus x^*} L(y^*) \right], \tag{12}$$

where

$$L(y^*) = \begin{cases} \displaystyle\sum_{a \in D \cap \tilde{X}} \big(\rho(D \setminus y^*, a) - \rho(D, a)\big) & \textit{for all } y^* \in F, \\ 0 & \textit{otherwise.}^{14} \end{cases} \tag{13}$$

When $y^* \in F$, the lower bound $L(y^*)$ can be interpreted as the total substitution toward y^*, since each term $\rho(D \setminus y^*, a) - \rho(D, a)$ in the summation captures the

[14] Note that if ρ is not RU-rationalizable, it is possible that $L(x^*) > 1 - \sum_{a \in D \cap \tilde{X}} \rho(D, a) - \sum_{y^* \in (D \cap X^*) \setminus x^*} L(y^*)$ which we may interpret as an empty set.

substitution from alternative a to y^*. These bounds (12) allow us to compare the relative desirability of unobservable alternatives.[15]

By comparing the bounds (12) and the bounds (11) from the outside option approach obtained in Remark 5, the improvement of the bounds can be simply summarized as

$$\sum_{y^* \in D \cap X^*} L(y^*).$$

This quantity can be interpreted as the total substitution towards unobservable alternatives in D.

To see how to obtain the result, first notice that the upper bounds can be obtained easily given the lower bounds.[16] To get the lower bounds assume that $x^* \in F$. Then, by Monotonicity, we have $\rho^*(D, y^*) \leq \rho^*(D \setminus x^*, y^*)$ for all $y^* \in (D \cap X^*) \setminus x^*$. Thus we have $\rho^*(D, x^*) + \sum_{a \in D \cap \tilde{X}} \rho(D, a) = 1 - \sum_{y^* \in (D \cap X^*) \setminus x^*} \rho^*(D, y^*) \geq 1 - \sum_{y^* \in (D \cap X^*) \setminus x^*} \rho^*(D \setminus x^*, y^*) = \sum_{a \in D \cap \tilde{X}} \rho(D \setminus x^*, a)$. It follows that $\rho^*(D, x^*) \geq \sum_{a \in D \cap \tilde{X}} (\rho(D \setminus x^*, a) - \rho(D, a)) \equiv L(x^*)$.

In the next remark, we demonstrate how to calculate the bounds by using the example in Table 1 in Section I.

Remark 7. *We will obtain bounds of unobservable choice frequencies from the choice set $\{b, c, d\}$. In the dataset ρ in Table 1, we observe $\rho(\{b, d\}, b) = 1/2 + \varepsilon, \rho(\{b, c\}, b) = 1/2$, and $\rho(\{b, c, d\}, b) = 1/3$ as alternative b is observable. From these we can calculate the lower bounds $L(c)$ and $L(d)$ as follows:*

$$\begin{aligned}
L(c) &= \rho(\{b, d\}, b) - \rho(\{b, c, d\}, b) = 1/2 + \varepsilon - 1/3 = 1/6 + \varepsilon, \\
L(d) &= \rho(\{b, c\}, b) - \rho(\{b, c, d\}, b) = 1/2 - 1/3 = 1/6.
\end{aligned}$$

Thus, we can also obtain upper bounds for alternatives c and d as $1 - \rho(\{b, c, d\}, b) -$

[15] When $\mathcal{D}$ only contains menus of the form D and $D \setminus y^*$ for $y^* \in X^*$ for some $D \subseteq X$, these bounds are actually tight. That is, they coincide with the RUM bounds derived in the next section. However, when the choice frequencies for other menus are available, Monotonicity and RUM have further implications.

[16] Suppose that we obtain the lower bound $L(y^*)$ for all $y^* \in D \cap X^*$. To get the upper bound, simply subtract the lower bounds from 1 to obtain $\rho^*(D, y^*) \leq 1 - \sum_{a \in D \cap \tilde{X}} \rho(D, a) - \sum_{x^* \in D \cap (X^* \setminus y^*)} L(x^*)$.

$L(d) = 1/2$ and $1-\rho(\{b,c,d\},b)-L(c) = 1/2-\varepsilon$, respectively. Thus, as observed in the introduction, the bounds for alternative c are $[1/6+\varepsilon, 1/2]$; the bounds for alternative d are $[1/6, 1/2-\varepsilon]$.

When $\varepsilon \geq 1/6$, the lower bound for alternative c is at least as large as the upper bound for alternative d, which suggests that alternative c is weakly more desirable than alternative d. Thus Remark 6 generalizes the analysis done in Section I.

B Bounds derived from RU rationality

In this section, we further improve the bounds by assuming the full RU rationality based on our characterization.

Proposition 2. *Suppose that ρ is RU-rationalizable. Suppose also that ρ^* is a complete dataset that is RU-rationalizable and coincides with ρ on $\mathcal{M}$. For any $(D,x) \notin \mathcal{M}$, the bounds for the unobservable choice frequency $\rho^*(D,x)$ are obtained by $[\underline{\rho}(D,x), \overline{\rho}(D,x)]$, where*

$$\overline{\rho}(D,x) = \max_{r \in \mathbf{R}_+^{\{(D\setminus x, D) | x \in D \in 2^X\}}} \sum_{E: E \supseteq D} r(E \setminus x, E), \tag{14}$$

$$\underline{\rho}(D,x) = \min_{r \in \mathbf{R}_+^{\{(D\setminus x, D) | x \in D \in 2^X\}}} \sum_{E: E \supseteq D} r(E \setminus x, E) \tag{15}$$

subject to the following constraints: $r(E\setminus x, E) = K(\rho, E, x)$ for all $(E,x) \in \mathcal{M}$ and, for all nonempty $E \subseteq X$,

$$\sum_{x: x\in E, (E,x)\notin\mathcal{M}} r(E\setminus x, E) - \sum_{y: y\notin E, (E\cup y, y)\notin \mathcal{M}} r(E, E\cup y) = 1\{E = X\} + \sum_{y: y\notin E, (E\cup y, y)\in\mathcal{M}} K(\rho, E\cup y, y) - \sum_{x:(E,x)\in\mathcal{M}} K(\rho, E, x). \tag{16}$$

Compared with obtaining the bounds directly from (P1), our bounds in Proposition 2 are computationally more efficient. This is because this problem can be seen as a transshipment problem, which is well known in the network-flow theory literature. One of the key properties of this problem is that it is a linear program whose constraint matrix is an *incidence matrix* as its coefficient.[17] For this specific problem, a practical polynomial-time algorithm, called the *network simplex algorithm*, can be applied. We refer readers to Orlin, Plotkin and Tardos (1993) and Orlin (1997) for further computational aspects of the algorithm. When $\mathcal{D} = 2^X \setminus \{\emptyset\}$, the bound (14) can be further simplified into (46) as shown in Section G.5 in the supplemental appendix.

Kitamura and Stoye (2019) also derive bounds on counterfactual choice probabilities. Although their bounds are theoretically equivalent to those in Proposition 2, they may be considerably more computationally demanding because they solve (P1) in its original form and thus do not exploit the network (incidence-matrix) structure that enables the efficiency gains of the network simplex algorithm.

C Application to Lottery Data

To test our methods, we take a complete dataset from the experiment conducted by McCausland et al. (2020) with all choice frequencies observable, then delete some of the choice frequencies to obtain an incomplete dataset. We apply our methods and then compare the bounds to the original choice frequencies. We find that the bounds obtained from our methods greatly improve on those from the outside option approach.

In the experiment, the authors fixed a set $X = \{0, 1, 2, 3, 4\}$ of five lotteries and asked 141 participants to choose one from each nonempty subset of X. Each participant made a decision six times for each choice set. We aggregate these choice frequencies to construct a complete dataset denoted by ρ. The lottery dataset is nearly RU-rationalizable but not exactly. As our method can be applied only to RU-rationalizable datasets, we first fit a RUM to the dataset to get a calibrated dataset that is close to the original one and is RU-rationalizable. The details of this procedure are in Subsection G.8 of the supplemental appendix. Then, we mask the choice probabilities of lotteries 0 and 1 and pretend not to observe them; in other words, we set $X^* = \{0, 1\}$, $\tilde{X} = \{2, 3, 4\}$, and $\mathcal{D} = 2^X \setminus \{\emptyset\}$.

[17] An incidence matrix is a matrix representing the node-arc structure of a network such that each column contains exactly one $+1$ and exactly one -1, and zeros elsewhere.

Under this setup, we will compute three types of bounds on the probabilities of lotteries 0 and 1 being chosen in a given choice set D that contains both lotteries. The first type of bounds is (11) based on the outside option approach: $\left[0, 1 - \sum_{x \in D \cap \{2,3,4\}} \rho(D, x)\right]$. Note that the bounds for lotteries 0 and 1 are identical, as the outside option approach provides no information to distinguish among unobservable alternatives. The bounds are shown as blue dotted intervals in Figure 4. We call these bounds *naive bounds.*

The second type of bounds is established in Remark 6 using Monotonicity. As demonstrated in Remark 7, the bounds can be calculated easily given the dataset ρ. These bounds are shown in purple for alternative 0 and in yellow for alternative 1 in Figure 4. We call these bounds *Monotonicity bounds.*

The third and the last type of bounds is the one that exploits the full implication of RUM and is computed by the linear program (14). These bounds are shown in red for alternative 0 and in green for alternative 1 in Figure 4. We call these bounds *RUM bounds.*

There are two key takeaways from the figure. First, our bounds (i.e., the RUM bounds and the Monotonicity bounds) are substantially tighter than the naive bounds, particularly when the choice set is large.[18] This suggests that our methods effectively leverage additional information about unobserved choice frequencies. In particular, for the choice set $\{0, 1, 2, 3, 4\}$, the RUM bounds and the Monotonicity bounds are extremely tight and closely approximate the true values.

Second, and more importantly, the RUM bounds for the choice frequency of alternative 1 are always higher than those of alternative 0. Therefore, we can conclude that in any RU-rationalization, the probability that alternative 1 is chosen is higher than the probability that alternative 0 is chosen in all menus except $D = \{0, 1\}$. A similar conclusion can be made for the Monotonicity bounds. On the other hand, in the outside option approach, the bounds for alternatives 1 and 0 are exactly identical. This difference indicates that our method exploits

[18]While the Monotonicity bounds for $\{0, 1\}$ are not close to the RUM bounds, this is only because the bounds derived in Remark 6 use Monotonicity in a simple way to aid the exposition. Indeed as mentioned in footnote 15, since there are additional menus beyond $\{0, 1\}$, $\{0\}$, and $\{1\}$, Monotonicity has further implications beyond the given bounds. If these are taken into account, the Monotonicity bounds become very close to the RUM bound even on $\{0, 1\}$.

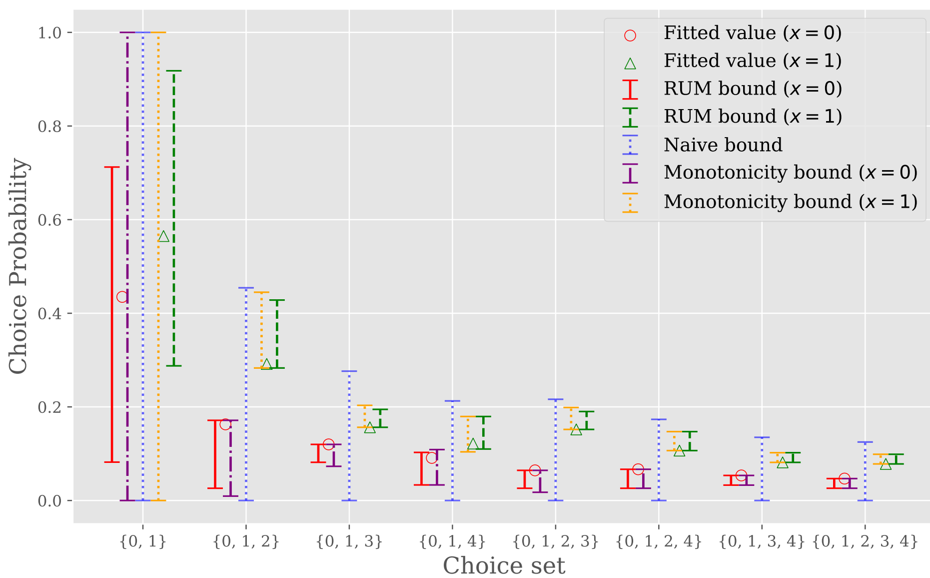

Figure 4: Comparison between the bounds of the choice probabilities of lotteries 0 and 1. The RUM bounds for 0 and 1 are shown in red and green, respectively. The Monotonicity bounds for 0 and 1 are shown in purple and yellow, respectively. The naive bounds for both are shown in blue.

the information that alternative 1 is better than alternative 0, while the outside option approach loses this information completely. This observation suggests that the estimation of the desirability of unobserved alternatives would be biased in the outside option approach, though it is beyond the scope of the current paper.

VI Concluding Remarks

In this paper, we consider a model of stochastic choice where some choice frequencies are unobservable. We formally demonstrate the limitations of the outside option approach, a prevalent approach to analyzing such datasets in empirical research. We begin by characterizing the full implications of the RUM when some choice frequencies are unobservable (Theorem 1), and then identify which implications are lost under the outside option approach (Proposition 1). Furthermore, in Remark 6 and Proposition 2, we develop methods to bound the unobserved choice frequencies, revealing information about the desirability of unobservable alternatives. This information is entirely lost in the outside option approach, as all unobservable alternatives are aggregated into a single alternative there. Taken together, these results underscore the value of accurately identifying the set of alternatives available to each agent. In particular, our results suggest that carefully measuring the availability of each outside alternative enables a more reliable inference of consumer preferences.

Appendix

A Proof of Lemma 2

We first prove that (P1) implies (P2). Fix a solution μ to (P1). Define a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ by $\rho^*(D,x) = \mu(\succ\in \mathcal{L} \mid x \succ y \text{ for all } y \in D\setminus x)$ for all $x \in D \in 2^X$. Then $\rho^* = \rho$ on $\mathcal{M}$. Moreover, by Lemma 1, we have $r \in \mathbf{R}_+^{\{(D\setminus x,D)|x\in D\in 2^X\}}$ that satisfies (5), (6) and $r(D\setminus x, D) = K(\rho^*, D, x)$ for all (D,x) such that $x \in D \in 2^X$. Since $\rho^* = \rho$ on $\mathcal{M}$, we have $r(D\setminus x, D) = K(\rho^*, D, x) = K(\rho, D, x)$ for all $(D,x) \in \mathcal{M}$. Thus, r is a solution to (P2).

Next we prove that (P2) implies (P1). Fix a solution r to (P2). For any (D,x) such that $x \in D \in 2^X$, define $\rho^*(D,x) \equiv \sum_{E:E\supseteq D} r(E\setminus x, E)$. Then $\rho^* = \rho$ on $\mathcal{M}$, where ρ is the given incomplete dataset. Moreover, by calculation, we have $\sum_{x\in D}\rho^*(D,x) = \sum_{x\in D}\sum_{E:E\supseteq D} r(E\setminus x,E) = \sum_{x\in X} r(X\setminus x, X) = 1$, where the first equality holds by (6) and the last equality holds by (5). Thus, we obtain a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$. (See the proof of Lemma 9 in Section F.5 of the supplemental appendix for the details of the calculation.)

Then by the Möbius inversion, we have $r(D\setminus x, D) = K(\rho^*, D, x)$ for all (D,x) such that $x \in D \in 2^X$. Then r satisfies (5), (6), and (7). By Lemma 1, there exists $\mu \in \Delta(\mathcal{L})$ such that $\rho^*(D,x) = \mu(\succ\in\mathcal{L} \mid x \succ y \text{ for all } y \in D\setminus x)$ for all (D,x) such that $x \in D \in 2^X$. Since $\rho = \rho^*$ on $\mathcal{M}$, (4) holds for any $(D,x)\in\mathcal{M}$. Thus, μ is a solution to (P1).

B Proof of Statement (a) of Theorem 1

To formalize the intuition explained in the main body of the paper, we introduce a definition:

Definition 7. *A collection $\mathcal{C} \subseteq 2^X$ is said to be complete if $D \in \mathcal{C} \implies \forall x \in X^*, D \cup x \in \mathcal{C}$.*

Remark 8. *If a collection $\mathcal{C} \subseteq 2^X$ has no unobservable outflows then it is complete. Moreover, if $\mathcal{C} \subseteq \mathcal{D}$ is complete then it has no unobservable outflows.*

We need one more definition: for any $\mathcal{C} \subseteq 2^X$, define

$$\delta_\rho(\mathcal{C}) \equiv \left(\sum_{\substack{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C},\\(D\cup x,x)\in\mathcal{M}}} K(\rho, D\cup x, x) - \sum_{\substack{(E,y):E\notin\mathcal{C},E\cup y\in\mathcal{C},\\(E\cup y,y)\in\mathcal{M}}} K(\rho, E\cup y, y) \right) \\ +1\{X\in\mathcal{C},\emptyset\notin\mathcal{C}\} - 1\{\emptyset\in\mathcal{C}, X\notin\mathcal{C}\}. \tag{17}$$

Remark 9. *For any $\mathcal{C} \subseteq 2^X$, $\delta_\rho(\mathcal{C})$ is the net observable outflow from $\mathcal{C}$. Therefore, if all inflows and outflows from $\mathcal{C}$ are observable then $\delta_\rho(\mathcal{C}) = 0$.*

To see this, notice that the first term is the values of the observable outflows from $\mathcal{C}$ and the second term is the values of the observable inflows to $\mathcal{C}$. Note also that the last indicator functions can be written simply as $+1\{X \in \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}\}$. See Section F.1 in the supplemental appendix for the proof.

Remark 10. *Suppose that $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$. If there are no observable inflows to $\mathcal{C}$ and $\emptyset \notin \mathcal{C}$, then $\delta_\rho(\mathcal{C}) \geq 0$.*

To see this notice that if there are no observable inflows and $\emptyset \notin \mathcal{C}$, then $\delta_\rho(\mathcal{C}) \geq \sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C},(D\cup x,x)\in\mathcal{M}} K(\rho, D\cup x, x)$, which is nonnegative by the nonnegativity of $K(\rho, D, x)$ for all $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$.[19]

The function δ_ρ satisfies disjoint additivity as follows:

[19]Note that since $\emptyset \notin \mathcal{C}$, we have $D \neq \emptyset$ for all $D \in \mathcal{C}$. Thus, we do not need $K(\rho, D, x) \geq 0$ for D such that $|D| = 1$.

Lemma 3. *For any $\mathcal{C} \subseteq 2^X$, $\delta_\rho(\mathcal{C}) = \sum_{D\in\mathcal{C}} \delta_\rho(\{D\})$.*

We use the above remarks and the lemma repeatedly throughout the proofs.

Based on the discussion in Section IV.B, especially inequality (10), if a solution to $(P2)$ exists, then $\delta_\rho(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$. The next result shows the converse also holds. Moreover, the lemma shows that we do not have to check all complete collections: it suffices to check all essential test collections belonging to $\mathcal{D}$.

Lemma 4. *Given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$, a solution to (P2) exists if and only if (i) $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$; and (ii) $\delta_\rho(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C} \subseteq \mathcal{D}$.*

The lemma proves statement (a) of Theorem 1.

B.1 Proof of Lemma 3

It suffices to show $\delta_\rho(\mathcal{C}\cup\mathcal{E}) = \delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$ for any disjoint sets $\mathcal{C}, \mathcal{E} \subseteq 2^X$. If there are no arcs connecting $\mathcal{C}$ and $\mathcal{E}$, the claim is immediate. Otherwise, it remains to verify that the values of flows on arcs connecting $\mathcal{C}$ and $\mathcal{E}$ cancel when $\delta_\rho(\mathcal{C})$ and $\delta_\rho(\mathcal{E})$ are summed.

Without loss of generality, suppose that there is a connecting arc $(D, D \cup x)$ and $D \in \mathcal{C}$ and $D \cup x \in \mathcal{E}$. Then the value $K(\rho, D \cup x, x)$ of the flow on the arc is added to $\delta_\rho(\mathcal{C})$ and subtracted from $\delta_\rho(\mathcal{E})$. Thus the value is canceled out in $\delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$. In the same way, the value $K(\rho, D \cup x, x)$ of the flow on the connecting arc $(D, D \cup x)$ and $D \in \mathcal{E}$ and $D \cup x \in \mathcal{C}$ will be canceled out. See Section F.2 of the supplemental appendix for a detailed calculation.

B.2 Proof of Lemma 4

To prove Lemma 4, we first provide a general lemma. Fix a network $(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is a nonempty finite set and $\mathcal{A} \subseteq \{(D, E) \mid D, E \in \mathcal{N}, D \neq E\}$. $\mathcal{N}$ is the set of nodes; $\mathcal{A}$ is the set of arcs. Consider a function $r : \mathcal{A} \to \mathbf{R}$. For any node $D \in \mathcal{N}$, let $r(D, \mathcal{N}) \equiv \sum_{E\in\mathcal{N}} r(D, E)$; $r(\mathcal{N}, D) \equiv \sum_{E\in\mathcal{N}} r(E, D)$[20].

[20]We define $r(D, \mathcal{N}) = 0$ if $(D, E) \notin \mathcal{A}$ for any $E \in \mathcal{N}$; Similarly, $r(\mathcal{N}, D) = 0$ if $(E, D) \notin \mathcal{A}$ for any $E \in \mathcal{N}$.

A function $r : \mathcal{A} \to \mathbf{R}$ is called a *flow* on a network $(\mathcal{N}, \mathcal{A})$ if it satisfies the following conditions:

$$r(s, \mathcal{N}) - r(\mathcal{N}, s) = 1, \tag{18}$$

$$r(D, \mathcal{N}) - r(\mathcal{N}, D) = 0 \quad \forall D \in \mathcal{N} \setminus \{s, t\}, \tag{19}$$

$$r(\mathcal{N}, t) - r(t, \mathcal{N}) = 1. \tag{20}$$

Notice that $r(D, \mathcal{N})$ is the sum of *outflows* from D, and that $r(\mathcal{N}, D)$ is the sum of *inflows* to D. Thus (18) means the net outflow from s is one; (19) means the inflows equal the outflows at each node $D \notin \{s, t\}$; (20) means the net inflow to t is one.

The following lemma provides a necessary and sufficient condition for the existence of a nonnegative flow satisfying some capacity constraints. For each arc (D, E), let $l(D, E)$ and $u(D, E)$ be exogenously given lower and upper bounds of the flow $r(D, E)$ of the arc. We prove the result using the maximum-flow theorem from Ford and Fulkerson (1962). We provide the proof in Section F.4 of the supplemental appendix.[21]

Lemma 5. *Let $l, u : \mathcal{A} \to \mathbf{R}_+ \cup \{+\infty\}$ be such that $l(D, E) \leq u(D, E)$ for $(D, E) \in \mathcal{A}$. There exists a flow $r : \mathcal{A} \to \mathbf{R}_+$ such that*

$$l(D, E) \leq r(D, E) \leq u(D, E) \quad \forall (D, E) \in \mathcal{A} \tag{21}$$

if and only if the following condition holds for all $\mathcal{C} \subseteq \mathcal{N}$

$$\sum_{(D,E) \in \mathcal{C} \times \mathcal{C}^c} u(D, E) - \sum_{(D,E) \in \mathcal{C}^c \times \mathcal{C}} l(D, E) \geq \begin{cases} 1 & \textit{if } t \notin \mathcal{C}, s \in \mathcal{C}, \\ -1 & \textit{if } t \in \mathcal{C}, s \notin \mathcal{C}, \\ 0 & \textit{otherwise.} \end{cases} \tag{22}$$

To interpret condition (22), remember that for any collection $\mathcal{C} \subseteq \mathcal{N}$, $r(D, E)$ is called an *outflow* from $\mathcal{C}$ if $D \in \mathcal{C}$ and $E \notin \mathcal{C}$; $r(D, E)$ is called an *inflow* to $\mathcal{C}$ if $D \notin \mathcal{C}$ and $E \in \mathcal{C}$. Thus the left-hand side is the sum of the upper bounds of

[21] We appreciate Professor Ui, who pointed out a similar result appears in Rockafellar (1998).

outflows from $\mathcal{C}$ minus the sum of the lower bounds of inflows to $\mathcal{C}$.[22] On the other hand, the right-hand side is the net outflow from $\mathcal{C}$.

We first prove the following lemma by using Lemma 5.

Lemma 6. *Given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$, a solution to (P2) exists if and only if (i) $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$; and (ii) $\delta_\rho(\mathcal{C}) \geq 0$ for any complete collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$.*

To apply Lemma 5 to our setup, let

$$\begin{aligned} &\mathcal{N} = 2^X, \quad \mathcal{A} = \{(D, D \cup x) \mid D \subseteq X, x \notin D\}, \; s = \emptyset, \quad t = X, \\ &l(D, D \cup x) = u(D, D \cup x) = K(\rho, D \cup x, x) \text{ if } (D \cup x, x) \in \mathcal{M}, \\ &l(D, D \cup x) = 0 \text{ and } u(D, D \cup x) = +\infty \;\text{ if } (D \cup x, x) \notin \mathcal{M}. \end{aligned} \tag{23}$$

Under the setup (23), there exists a solution r to (P2) $\Leftrightarrow$ there exists a non-negative flow r that satisfies the conditions in Lemma 5 $\Leftrightarrow$ (i) $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$; and (ii) the condition (22) holds for any $\mathcal{C} \subseteq \mathcal{N}$, where the first equivalence holds by the setup and the second equivalence holds by Lemma 5. Thus, to show Lemma 6, it suffices to prove that the condition (22) holds for any $\mathcal{C} \subseteq \mathcal{N}$ if and only if $\delta_\rho(\mathcal{C}) \geq 0$ for any complete collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$. We prove this equivalence in Section F.3 of the supplemental appendix.

Given Lemma 6, we prove Lemma 4 by proving the following two lemmas. The first lemma shows that checking all test collections belonging to $\mathcal{D}$, rather than all complete collections, is enough.

Lemma 7. *Suppose that $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$. If $\delta_\rho(\mathcal{C}) \geq 0$ for any test collection $\mathcal{C} \subseteq \mathcal{D}$, then $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$.*

The next lemma shows that we do not have to check $\delta_\rho(\mathcal{C}) \geq 0$ for the nonessential test collections.

[22] In network-flow theory, this value is called residual capacity of a cut $(\mathcal{C}, \mathcal{C}^c)$.

Lemma 8. *Let $\mathcal{C} \equiv \{A \cup E \mid E \in \mathcal{E}\}$ be a test collection with $A \subseteq \tilde{X}$ and $\mathcal{E} \subseteq 2^{X^*}$. Assume that $\mathcal{C} \subseteq \mathcal{D}$. (i) If $\mathcal{E} = 2^{X^*}$, then $\delta_\rho(\mathcal{C}) = 0$; (ii) Suppose $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$. If $A = \tilde{X}$ or $A = \emptyset$, then $\delta_\rho(\mathcal{C}) \geq 0$.*

Combining Lemma 6, 7, and 8 immediately implies Lemma 4.

B.2.1 Proof of Lemma 7

We will prove the statement by the following two steps.

Step 1: If $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any test collection $\hat{\mathcal{C}} \subseteq \mathcal{D}$, then $\delta_\rho(\mathcal{C}) \geq 0$ for any test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$.

Proof. Fix a test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$ to show that $\delta_\rho(\mathcal{C}) \geq 0$. If $\mathcal{C} \subseteq \mathcal{D}$, then by the supposition of the step, we have $\delta_\rho(\mathcal{C}) \geq 0$. In the following, we assume that $\mathcal{C} \nsubseteq \mathcal{D}$.

Case 1: Suppose that $\mathcal{C} \cap \mathcal{D} = \emptyset$. Since $\mathcal{D}$ is an upper set (i.e., $D \in \mathcal{D}$ & $E \supseteq D \implies E \in \mathcal{D}$), $\mathcal{C} \cap \mathcal{D} = \emptyset$ means that there are no observable inflows into $\mathcal{C}$. Since $\emptyset \notin \mathcal{C}$, Remark 10 yields $\delta_\rho(\mathcal{C}) \geq 0$.

Case 2: Suppose that $\mathcal{C} \cap \mathcal{D} \neq \emptyset$. Let $\mathcal{C}^* = \mathcal{C} \cap \mathcal{D}$. Since $\mathcal{D}$ is an upper set, $\mathcal{D}$ is complete. Since $\mathcal{C}$ is also complete, their intersection $\mathcal{C}^*$ is complete; thus a test collection. Since $\mathcal{C}^* \subseteq \mathcal{D}$, it follows from our supposition that $\delta_\rho(\mathcal{C}^*) \geq 0$. By disjoint additivity of δ_ρ (Lemma 3), $\delta_\rho(\mathcal{C}) = \delta_\rho(\mathcal{C}^*) + \sum_{D \in \mathcal{C} \setminus \mathcal{C}^*} \delta_\rho(\{D\})$. Since for any $D \in \mathcal{C} \setminus \mathcal{C}^* \equiv \mathcal{C} \cap \mathcal{D}^c$, we have $D \notin \mathcal{D}$. Since $D \neq \emptyset$, by Remark 10, we have $\delta_\rho(\{D\}) \geq 0$. Thus, we have $\delta_\rho(\mathcal{C}) \geq 0$. ■

Step 2: If $\delta_\rho(\mathcal{C}) \geq 0$ for any test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$, then $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$.

Proof. Fix a complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$. Decompose $\hat{\mathcal{C}}$ as follows: for each $A \subseteq \tilde{X}$ write $\mathcal{C}_A = \{D \in \hat{\mathcal{C}} : D \setminus X^* = A\}$. Clearly $\hat{\mathcal{C}} = \bigcup_{A \subseteq \tilde{X}} \mathcal{C}_A$.

It is easy to see that each nonempty $\mathcal{C}_A$ is a test collection and $\emptyset \notin \mathcal{C}_A$. Thus by the assumption of the step, $\delta_\rho(\mathcal{C}_A) \geq 0$. Notice that for $A \neq B$, $\mathcal{C}_A$ and $\mathcal{C}_B$ are disjoint. By Lemma 3, $\delta_\rho(\hat{\mathcal{C}})$ can be written as $\delta_\rho(\hat{\mathcal{C}}) = \sum_{A \subseteq \tilde{X}} \delta_\rho(\mathcal{C}_A) \geq 0$. ■

B.2.2 Proof of Lemma 8

Let $\mathcal{C} \equiv \{A \cup E \mid E \in \mathcal{E}\}$ be a test collection with $A \subseteq \tilde{X}$ and $\mathcal{E} \subseteq 2^{X^*}$. Assume that $\mathcal{C} \subseteq \mathcal{D}$.

Step 1: If $\mathcal{E} = 2^{X^*}$, then $\delta_\rho(\mathcal{C}) = 0$.

Proof. By the fact that $\mathcal{E} = 2^{X^*}$ and $\mathcal{C} \subseteq \mathcal{D}$, all flows into and out of $\mathcal{C}$ are observable.[23] By Remark 9, it follows that $\delta_\rho(\mathcal{C})$ is zero. ■

Step 2: Suppose $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ s.t. $1 < |D| < |X|$. If $A = \tilde{X}$ or $A = \emptyset$, then $\delta_\rho(\mathcal{C}) \geq 0$.

Proof. Assume $A = \tilde{X}$. First we show that $\delta_\rho(\{D\}) \leq 0$ for all $D \in \mathcal{C} \setminus X$. Fix any $D \in \mathcal{C} \setminus X$. Since $A = \tilde{X}$, there are no observable flows coming out from $\{D\}$. Since $|D| \geq |\tilde{X}| > 1$ and $D \neq X$, $K(\rho, D, x) \geq 0$ for all $x \in \tilde{X}$; it follows from the definition of δ_ρ that $\delta_\rho(\{D\}) \leq 0$.

We now show that $\delta_\rho(\mathcal{C}) \geq 0$. Remember $\delta_\rho(\mathcal{C}) = \sum_{D \in \mathcal{C}} \delta_\rho(\{D\})$. Since $\mathcal{C}$ is complete, $A = \tilde{X}$ implies $X = \tilde{X} \cup X^* \in \mathcal{C}$. Since $\delta_\rho(\{D\}) \leq 0$ for all $D \in \mathcal{C} \setminus X$ and $X \in \mathcal{C}$, it suffices to prove $\delta_\rho(\mathcal{C}) = 0$ where $\mathcal{C}$ is the largest, or $\mathcal{C} = \{\tilde{X} \cup E \mid E \in 2^{X^*}\}$. Note $\mathcal{C} \subseteq \mathcal{D}$ as $\tilde{X} \in \mathcal{D}$. By Step 1, we have $\delta_\rho(\{\tilde{X} \cup E \mid E \in 2^{X^*}\}) = 0$.

Assume $A = \emptyset$. If $\emptyset \in \mathcal{C}$, then $\mathcal{C} = 2^{X^*}$ since $\mathcal{C}$ is complete; thus $\delta_\rho(\mathcal{C}) = 0$ by Remark 9. If $\emptyset \notin \mathcal{C}$, $\delta_\rho(\mathcal{C}) \geq 0$ by Remark 10. ■

C Proof of Statement (b) of Theorem 1

We prove the following two statements, which imply statement (b).

(1) For each $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$, there exists an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$ such that (a) $K(\rho, D, x) < 0$; and $K(\rho, E, y) \geq 0$ for all $(E, y) \in \mathcal{M} \setminus (D, x)$; (b) $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collections $\mathcal{C} \subseteq \mathcal{D}$.

(2) For each essential test collection $\mathcal{C}^* \subseteq \mathcal{D}$, there exists an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$ such that (a) $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$;(b) $\delta_\rho(\mathcal{C}^*) < 0$ and $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collections $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$.

[23] That is, (i) if there exists (D, x) such that $D \in \mathcal{C}$ and $D \cup x \notin \mathcal{C}$, then $D \in \mathcal{D}$ and $x \in \tilde{X}$; and (ii) if there exists (E, y) such that $E \notin \mathcal{C}$ and $E \cup y \in \mathcal{C}$, then $E \cup y \in \mathcal{D}$ and $y \in \tilde{X}$.

To prove the two statements above, we show the corresponding statements in terms of flows assuming $\mathcal{D} = 2^X \setminus \{\emptyset\}$ in Lemma 10 below. Then, we convert the flows into a complete dataset by using the next lemma:

Lemma 9. *Let $\mathcal{D} = 2^X \setminus \{\emptyset\}$. Let $r \in \mathbf{R}^{\{(D \setminus x, D) | x \in D \in 2^X\}}$. The vector r satisfies the following three conditions:*

(i) $\sum_{x \in X} r(X \setminus x, X) = 1$;

(ii) for any $D \in \mathcal{D}$ s.t. $1 \leq |D| < |X|$, $\sum_{x \in D} r(D \setminus x, D) = \sum_{y \notin D} r(D, D \cup y)$;

(iii) for any $x \in D \in \mathcal{D}$, $\sum_{E: E \supseteq D} r(E \setminus x, E) \geq 0$,

if and only if there exists a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x) | x \in D \in 2^X\}}$ such that $\sum_{x \in D} \rho^*(D, x) = 1$ for all $D \in D$ and $K(\rho^*, D, x) = r(D \backslash x, D)$ for any (D, x) such that $x \in D \in 2^X$.

We provide the proof of the lemma in Section F.5 in the supplemental appendix.

We now introduce a new notation δ_r that corresponds to the definition (17) of δ_ρ: for any $\mathcal{C} \subseteq \mathcal{N}$, define[24]

$$\delta_r(\mathcal{C}) = \left(\sum_{\substack{(D,x): \\ D \in \mathcal{C}, D \cup x \notin \mathcal{C}, x \in \tilde{X}}} r(D, D \cup x) - \sum_{\substack{(E,y): \\ E \notin \mathcal{C}, E \cup y \in \mathcal{C}, y \in \tilde{X}}} r(E, E \cup y) \right) + 1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}. \tag{24}$$

Lemma 10. *Let $\mathcal{D} = 2^X \setminus \{\emptyset\}$.*

(i) For each $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$, there exists $r^* \in \mathbf{R}^{\{(D \setminus x, D) | x \in D \in \mathcal{D}\}}$ that satisfies conditions (i)–(iii) in Lemma 9 and the following two conditions: (a) $r^*(D \setminus x, D) < 0$; $r^*(E \setminus y, E) \geq 0$ for all $(E, y) \in \mathcal{M} \setminus (D, x)$; (b) $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.

[24] Conditions after the summations are different between δ_ρ and δ_r because we are considering the complete dataset here. When $\mathcal{D} = 2^X \setminus \{\emptyset\}$, both conditions are equivalent.

(ii) For each essential test collection $\mathcal{C}^*$, there exists $r^* \in \mathbf{R}^{\{(D\setminus x,D)|x\in D\in\mathcal{D}\}}$ that satisfies conditions (i)–(iii) in Lemma 9 and the following two conditions: (a) $r^*(D \setminus x, D) \geq 0$ for all $x \in \tilde{X}$; (b) $\delta_{r^*}(\mathcal{C}^*) < 0$ and $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$ except $\mathcal{C}^*$.

We will provide the proof of Lemma 10 in the next section. Given Lemma 9 and 10, we now prove statement (2) stated at the beginning of the section. Fix $\mathcal{C}^* \subseteq \mathcal{D}$. By Lemma 10 (ii), there exists $r^* \in \mathbf{R}^{\{(D\setminus x,D)|x\in D\in 2^X\}}$ that satisfies conditions (i)–(iii) in Lemma 9 and conditions (a) and (b) in Lemma 10 (ii). By Lemma 9, there exists a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ such that $\sum_{x\in D}\rho^*(D,x) = 1$ for all $D \in 2^X \setminus \{\emptyset\}$ and $K(\rho^*, D, x) = r^*(D \setminus x, D)$ for any (D, x) such that $x \in D \in 2^X$. Let ρ be the restriction of ρ^* to $\mathcal{M}$. Since $\mathcal{D}$ is an upper set, $K(\rho, D, x) = K(\rho^*, D, x)$ for all $(D, x) \in \mathcal{M}$. In the following, we will show that $\delta_\rho(\mathcal{C}^*) < 0$ and $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collections $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$. When $\mathcal{C} \subseteq \mathcal{D}$, then $\delta_{\rho^*}(\mathcal{C})$ only depends on the values of ρ^* in $\mathcal{M}$. Since $\rho = \rho^*$ on $\mathcal{M}$, $K(\rho, D, x) = r^*(D \setminus x, D)$ when $(D, x) \in \mathcal{M}$. Thus, for any test collection $\mathcal{C} \subseteq \mathcal{D}$, we have $\delta_{r^*}(\mathcal{C}) = \delta_\rho(\mathcal{C})$. Consequently, we have $\delta_\rho(\mathcal{C}^*) < 0$ and $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collections $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$. Statement (1) can be proved exactly in the same way using Lemma 10 (i) instead of (ii).

C.1 Proof of Statement (i) in Lemma 10

Let $\mathcal{H} \equiv \{D \in \mathcal{D} \mid \text{there exists an essential test collection } \mathcal{C} \text{ such that } D \in \mathcal{C}\}$. Explicitly, $\mathcal{H}$ is the set of all $D \in \mathcal{D}$ which have nonempty intersection with X^*, nonempty intersection with $\tilde{X}$, and do not contain $\tilde{X}$.

Remark 11. *For any $r \in \mathbf{R}_+^{\{(D\setminus x,D)|x\in D\in\mathcal{D}\}}$ and any essential test collection $\mathcal{C}$, if $\delta_r(\{D\}) \geq 0$ for all $D \in \mathcal{H}$, then $\delta_r(\mathcal{C}) \geq 0$.*

To see the remark, notice that δ_r is disjoint additive which can be seen by applying the same argument as Lemma 3 to the case $\mathcal{D} = 2^X \setminus \{\emptyset\}$. Thus, for all $\mathcal{C} \subseteq \mathcal{H}$, $\delta_r(\mathcal{C}) = \sum_{D\in\mathcal{C}} \delta_r(\{D\}) \geq 0$ since $\delta_r(\{D\}) \geq 0$ for all $D \in \mathcal{H}$.

For any directed path Π, denote by r^Π the flow which puts weight one on all arcs in Π and zero otherwise.

To prove statement (i), let $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$. Then, $x \in \tilde{X}$, $D \setminus x \neq \emptyset$, and $D \neq X$ hold. We will consider two cases:

Case 1: $\tilde{X} \not\subseteq D$. Let Π_1 be a dipath from $\emptyset$ to D which passes through $D \cap \tilde{X}$.[25] Let Π_2 be a $D \setminus x$ to X dipath which passes through $D \cup \tilde{X}$ but not D. Such a dipath exists because $\tilde{X} \not\subseteq D$ implies that there exists an observable alternative $y \in D^c \cap \tilde{X}$ and there exists an arc $(D \setminus x, (D \setminus x) \cup y)$ such that $(D \setminus x) \cup y \neq D$. Define

$$r^* \equiv r^{\Pi_1} - r^{(D\setminus x, D)} + r^{\Pi_2},$$

where $r^{(D\setminus x,D)}$ is a vector that gives one only at the arc $(D \setminus x, D)$; zero elsewhere. See Figure 5 for the illustration of the flow r^*. The flow r^* satisfies the conditions of Lemma 9: conditions (i) and (ii) are satisfied trivially; the condition (iii) is also satisfied because the negative value on $(D \setminus x, D)$ is canceled by one of the positive values on Π_2. In particular, Π_2 contains an arc of the form $(D \cup E \setminus x, D \cup E)$ for some $E \subseteq \tilde{X}$.

By definition, $r^*(D \setminus x, D) = -1$ and $r^*(E \setminus y, E) \geq 0$ for any (E, y) such that $y \in \tilde{X}$ and $(D, x) \neq (E, y)$, which ensures that condition (a) is satisfied. To see condition (b), notice first that by the definition of r^*, we have $\delta_{r^*}(\{D \setminus x\}) = 0$ and $\delta_{r^*}(\{D\}) \geq 0$. For all other nodes $E \in \mathcal{H}$, $\delta_{r^*}(\{E\}) = 0$. Thus, we have $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.

Case 2: $\tilde{X} \subseteq D$. Notice $D \setminus x \neq (D \setminus x) \cup X^*$ since otherwise $D \supseteq X^*$, which implies $D = X$, a contradiction to the assumption $D \neq X$. So let Π_3 be a $D \setminus x$ to X dipath that passes through $(D \setminus x) \cup X^*$. (Note that the last arc is an observable arc $(X \setminus x, X)$, where $x \in \tilde{X}$.) Let Π_1 be defined as in Case 1 and define

$$r^* \equiv r^{\Pi_1} - r^{(D\setminus x, D)} + r^{\Pi_3}.$$

The flow r^* satisfies the conditions of Lemma 9 as in Case 1. Moreover, by definition, $r^*(D \setminus x, D) < 0$ and $r^*(E \setminus y, E) \geq 0$ for any (E, y) such that $y \in \tilde{X}$ and $(D, x) \neq (E, y)$. In addition, for any essential test collection $\mathcal{C}$, $\delta_{r^*}(\mathcal{C}) \geq 0$.

[25] Π_1 is chosen not to pass through $(D \setminus x, D)$.

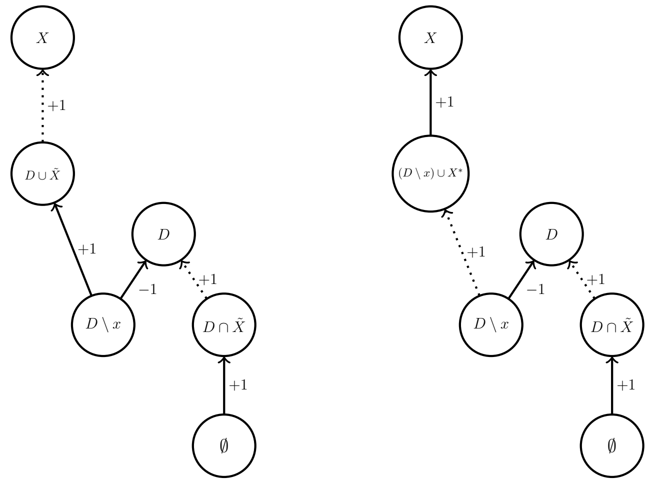

Figure 5: Flow $r^{\Pi_1} - r^{(D \setminus x, D)} + r^{\Pi_2}$ in Case 1 (left); Flow $r^{\Pi_1} - r^{(D \setminus x, D)} + r^{\Pi_3}$ in Case 2 (right) of Section C.1. Given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$, solid arrows correspond to observable flows and dotted arrows correspond to unobservable flows. Note also that all nodes other than D, $D \setminus x$ and $(D \setminus x) \cup X^*$ appearing in the figure do not belong to $\mathcal{H}$; thus the values of δ_{r^*} are zero on $\mathcal{H} \setminus \{D, D \setminus x, (D \setminus x) \cup X^*\}$.

To see this notice (i) $\delta_{r^*}(\{D \setminus x\}) = -1$ but $\delta_{r^*}(\{(D \setminus x) \cup X^*\}) = 1$; (ii) any test collection $\mathcal{C}$ containing $D \setminus x$ contains $(D \setminus x) \cup X^*$. Statements (i) and (ii) imply that the negative value of $\delta_{r^*}(\{D \setminus x\})$ is cancelled by the positive value of $\delta_{r^*}(\{(D \setminus x) \cup X^*\})$. For all other nodes $E \in \mathcal{H}$, $\delta_{r^*}(\{E\}) = 0$. Thus, we have $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.

C.2 Proof of Statement (ii) of Lemma 10

Fix an essential test collection $\mathcal{C}^* = \{A^* \cup E \mid E \in \mathcal{E}^*\}$ where $\emptyset \neq A^* \subsetneq \tilde{X}$ and $\mathcal{E}^* \subsetneq 2^{X^*}$ is a nonempty upper set. We will construct a flow r^* such that $\delta_{r^*}(\mathcal{C}^*) < 0$ and $\delta_{r^*}(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$.

We first state the following claims. The proofs are in Section F.6 and F.7 of the supplemental appendix.

Remember that $\mathcal{H} \equiv \{D \in \mathcal{D} \mid \exists$ an essential test collection $\mathcal{C}$ s.t. $D \in \mathcal{C}\}$.

Claim 1. *There exists a $\emptyset - X$ directed path Π_4 avoiding all nodes in $\mathcal{H}$.*

Claim 2. *Fix $\varepsilon \in (0, 1]$. For any D in $\mathcal{H}$, there exist flows r_D^1 and r_D^2 such that $\delta_{r_D^1}(\{D\}) = -\varepsilon$ and $\delta_{r_D^1}(\{H\}) = 0$ for any $H \in \mathcal{H} \setminus \{D\}$; $\delta_{r_D^2}(\{D\}) = \varepsilon$ and*

$\delta_{r^2_D}(\{H\}) = 0$ *for all* $H \in \mathcal{H} \setminus \{D\}$. *Moreover* r^1_D *and* r^2_D *are nonnegative on observable arcs and satisfy the three conditions in Lemma 9.*

To construct r^*, define:

- $r^3 = \sum_{D \in \mathcal{C}^*} \frac{1}{|\mathcal{C}^*|} r^1_D$, where r^1_D is from Claim 2;
- $r^4 = \frac{1}{|\mathcal{C}^*|} r^{\Pi_4} + \frac{|\mathcal{C}^*|-1}{|\mathcal{C}^*|} r^2_{A^* \cup X^*}$, where Π_4 is from Claim 1 and $r^2_{A^* \cup X^*}$ is from Claim 2;
- $\hat{r} = \frac{1}{2} r^3 + \frac{1}{2} r^4$.

By construction, $\delta_{\hat{r}}(\{D\}) = -\frac{\varepsilon}{2|\mathcal{C}^*|}$ for $D \in \mathcal{C}^* \setminus \{A^* \cup X^*\}$, $\delta_{\hat{r}}(\{A^* \cup X^*\}) = -\frac{\varepsilon}{2|\mathcal{C}^*|} + \frac{|\mathcal{C}^*|-1}{2|\mathcal{C}^*|}\varepsilon$, and $\delta_{\hat{r}}(\{H\}) = 0$ for $H \in \mathcal{H} \setminus \mathcal{C}^*$. This gives $\delta_{\hat{r}}(\mathcal{C}^*) = -\frac{\varepsilon}{2|\mathcal{C}^*|} < 0$.

For each essential test collection $\mathcal{C} = \{A^* \cup E : E \in \mathcal{E}\}$ for some upper set $\mathcal{E} \subseteq 2^{X^*}$ such that $\mathcal{C} \supsetneq \mathcal{C}^*$, choose an element of $\mathcal{C} \setminus \mathcal{C}^*$ arbitrarily and denote it by $D_{\mathcal{C}}$. Let $\mathcal{F}$ be the collection of such $D_{\mathcal{C}}$. If $\mathcal{F} = \emptyset$ then $r^* = \hat{r}$, otherwise, define

$$r^* = \alpha \hat{r} + (1-\alpha) \sum_{D_{\mathcal{C}} \in \mathcal{F}} \frac{1}{|\mathcal{F}|} r^2_{D_{\mathcal{C}}},$$

where $\alpha \in (0,1)$ and $\alpha < \frac{2|\mathcal{C}^*|}{|\mathcal{F}|+2|\mathcal{C}^*|}$. The flow r^* satisfies the three conditions of Lemma 9, which follows because $\hat{r}$ and $r^2_{D_{\mathcal{C}}}$ satisfy them.

Let $\mathcal{C}$ be an arbitrary essential test collection with $\mathcal{C} = \{A \cup E \mid E \in \mathcal{E}\}$ and $\mathcal{C} \neq \mathcal{C}^*$. We consider the following three cases to show that $\delta_{r^*}(\mathcal{C}) \geq 0$.

Case 1: $A \neq A^*$**.** Then $\mathcal{C} \cap (\mathcal{C}^* \cup \mathcal{F}) = \emptyset$, so by disjoint additivity and the fact that $\delta_{r^*}(\{G\}) = 0$ for all $G \in \mathcal{H} \setminus (\mathcal{C}^* \cup \mathcal{F})$, we have $\delta_{r^*}(\mathcal{C}) = 0$.

Case 2: $A = A^*$ **and** $\mathcal{C} \not\supseteq \mathcal{C}^*$ **(i.e.,** $\mathcal{C}^* \setminus \mathcal{C} \neq \emptyset$**).** Since $\mathcal{C} \not\supseteq \mathcal{C}^*$, there exists $D^* \in \mathcal{C}^* \setminus \mathcal{C}$. Note that since both $\mathcal{C}$ and $\mathcal{C}^*$ are essential test collections with observable part $A = A^*$, and both are upper sets, we must have $A^* \cup X^* \in \mathcal{C} \cap \mathcal{C}^*$. By disjoint additivity, $\delta_{r^*}(\mathcal{C}) + \delta_{r^*}(\mathcal{C}^* \setminus (\mathcal{C} \cup \{D^*\})) + \delta_{r^*}(\{D^*\}) = \delta_{r^*}(\mathcal{C}^*) + \delta_{r^*}(\mathcal{C} \setminus \mathcal{C}^*)$. Notice that $\mathcal{C} \setminus \mathcal{C}^* \subseteq \mathcal{H} \setminus C^*$ implies $\delta_{r^*}(\mathcal{C} \setminus \mathcal{C}^*) \geq 0$, and that $\mathcal{C}^* \setminus (\mathcal{C} \cup \{D^*\}) \subseteq \mathcal{C}^* \setminus \{A^* \cup X^*\}$ implies $\delta_{r^*}(\mathcal{C}^* \setminus (\mathcal{C} \cup \{D^*\})) \leq 0$.[26] Since $D^* \in \mathcal{C}^* \setminus \mathcal{C}$, we have

[26] The only negative contribution to δ_{r^*} on $\mathcal{H}$ comes from r^1_D for $D \in \mathcal{C}^*$. Thus on $\mathcal{H} \setminus \mathcal{C}^*$, δ_{r^*} is nonnegative. Furthermore, inside of $\mathcal{C}^*$ the only positive contribution to δ_{r^*} is $r^2_{A^* \cup X^*}$ so that δ_{r^*} is nonpositive on $\mathcal{C}^* \setminus \{A^* \cup X^*\}$.

$D^* \neq A^* \cup X^*$; and thus $\delta_{r^*}(\{D^*\}) = \alpha\delta_{\hat{r}}(\{D^*\}) = -\frac{\varepsilon\alpha}{2|\mathcal{C}^*|}$. Recall also that $\delta_{r^*}(\mathcal{C}^*) = \alpha\delta_{\hat{r}}(\mathcal{C}^*) = -\frac{\varepsilon\alpha}{2|\mathcal{C}^*|}$. We obtain

$$\delta_{r^*}(\mathcal{C}) = \delta_{r^*}(\mathcal{C}^*)+\delta_{r^*}(\mathcal{C}\setminus\mathcal{C}^*)-\delta_{r^*}(\mathcal{C}^*\setminus(\mathcal{C}\cup\{D^*\}))-\delta_{r^*}(\{D^*\}) \geq -\frac{\varepsilon\alpha}{2|\mathcal{C}^*|}+\frac{\varepsilon\alpha}{2|\mathcal{C}^*|} = 0.$$

Case 3: $A = A^*$ and $\mathcal{C} \supsetneq \mathcal{C}^*$. Then $\mathcal{C} \cap \mathcal{C}^* = \mathcal{C}^*$ and there exists $D_{\mathcal{C}} \in \mathcal{C} \setminus \mathcal{C}^*$ (since $\mathcal{C} \supsetneq \mathcal{C}^*$). By construction, $r^2_{D_{\mathcal{C}}}$ contributes positively to $\delta_{r^*}(\mathcal{C})$. Specifically: $\delta_{r^*}(\mathcal{C}) \geq \alpha\delta_{\hat{r}}(\mathcal{C}^*) + (1-\alpha)\cdot\frac{1}{|\mathcal{F}|}\cdot\varepsilon = -\frac{\alpha\varepsilon}{2|\mathcal{C}^*|} + \frac{(1-\alpha)}{|\mathcal{F}|}\varepsilon$. This is nonnegative because $\frac{\alpha}{2|\mathcal{C}^*|} \leq \frac{1-\alpha}{|\mathcal{F}|}$.

D Proof of Propositions

D.1 Proof of Proposition 1

Suppose ρ satisfies the conditions of Proposition 1, that is ρ satisfies (i) and (ii) for singleton test collections. By McFadden (2005), it suffices to show that $K(\hat{\rho}, \hat{D}, x) \geq 0$ for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $1 < |\hat{D}| < |\tilde{X}| + 1$.

Remember that for all $\hat{D} \in \hat{\mathcal{D}}$, we write $D = (\hat{D} \cap \tilde{X}) \cup X^*$ if $x_0 \in \hat{D}$ and $D = \hat{D}$ otherwise. For all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $x \in \tilde{X}$, let $r(\hat{D}, x) = K(\rho, D, x)$ if $x_0 \in \hat{D}$ and $r(\hat{D}, x) = \sum_{G \subsetneq X^*} K(\rho, D \cup G, x)$ if $x_0 \notin \hat{D}$.

To prove the proposition, we use the following identity: for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $x \in \tilde{X}$,

$$\sum_{E \supseteq D} K(\rho, E, x) = \sum_{\hat{E} \supseteq \hat{D}} r(\hat{E}, x).$$

See Section F.8 of the supplemental appendix for the proof.

We first show the following claim:

Claim 3. *$r(\hat{D}, x) = K(\hat{\rho}, \hat{D}, x)$ for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $x \in \tilde{X}$.*

Proof. By definition we have $\hat{\rho}(\hat{D}, x) = \rho(D, x)$, thus $\hat{\rho}(\hat{D}, x) = \sum_{E \supseteq D} K(\rho, E, x)$ for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $x \in \tilde{X}$. By the identity above, therefore, $\hat{\rho}(\hat{D}, x) = \sum_{\hat{E} \supseteq \hat{D}} r(\hat{E}, x)$ for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $x \in \tilde{X}$.

Thus by the Möbius inversion, for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $x \in \tilde{X}$, $r(\hat{D}, x) = \sum_{\hat{E}:\hat{E} \supseteq \hat{D}} \hat{\rho}(\hat{E}, x)(-1)^{|\hat{E} \setminus \hat{D}|} = K(\hat{\rho}, \hat{D}, x)$. ■

We now show that $K(\hat{\rho}, \hat{D}, x) \geq 0$ for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$ and $1 < |\hat{D}| < |\tilde{X}| + 1$ by cases.

Case 1: $x \in \tilde{X}$. By Claim, $K(\hat{\rho}, \hat{D}, x) = r(\hat{D}, x)$. Since $r(\hat{D}, x)$ is nonnegative by condition (i) of Theorem 1, we are done.

Case 2: $x = x_0$. Fix $\hat{D}$ with $x_0 \in \hat{D}$. By the inflow-outflow equality, $\sum_{y \notin \hat{D}} K(\hat{\rho}, \hat{D} \cup y, y) = \sum_{z \in \hat{D}} K(\hat{\rho}, \hat{D}, z)$. It follows that $K(\hat{\rho}, \hat{D}, x_0) = \sum_{y \notin \hat{D}} K(\hat{\rho}, \hat{D} \cup y, y) - \sum_{z \in \hat{D} \cap \tilde{X}} K(\hat{\rho}, \hat{D}, z) = \sum_{y \notin D} K(\rho, D \cup y, y) - \sum_{z \in D \cap \tilde{X}} K(\rho, D, z) \geq 0$, where the first equality is obtained by rearranging the inflow-outflow equality, and the second equality follows by Claim and the definition of r with $x_0 \in \hat{D}$. Nonnegativity follows from condition (ii) in the case $\mathcal{C} = \{D\}$. Note that $\{D\}$ is an essential test collection because $D \supseteq X^*$ and $\emptyset \subsetneq D \cap \tilde{X} \subsetneq \tilde{X}$.

D.2 Proof of Proposition 2

Denote the set $\{(D, x) \mid x \in D \in 2^X\}$ by $\mathcal{M}^*$. Fix an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$. Let Γ be the set of RU-rationalizable complete datasets $\rho^* \in \mathbf{R}_+^{\mathcal{M}^*}$ that are extensions of ρ, i.e.,

$$\Gamma \equiv \left\{ \rho^* \in \mathbf{R}_+^{\mathcal{M}^*} \;\middle|\; \begin{array}{l} (\alpha)\ \rho = \rho^* \text{ on } \mathcal{M} \\ (\beta)\ \exists \mu \in \Delta(\mathcal{L}) \text{ s.t. } \forall (D, x) \in \mathcal{M}^*,\ \rho^*(D, x) = \mu\left(\succ \in \mathcal{L} \mid \forall y \in D \setminus x, x \succ y\right) \end{array} \right\}.$$

With $(D, x) \in \mathcal{M}^* \setminus \mathcal{M}$ fixed, the goal is to obtain bounds of $\rho^*(D, x)$ subject to $\rho^* \in \Gamma$. As pointed out by Manski (2007), since Γ is convex and conditions (α) and (β) are linear, the identified set of $\rho^*(D, x)$ is an interval with the upper and lower bounds given by $\overline{\rho}(D, x) \equiv \max_{\rho^* \in \Gamma} \rho^*(D, x)$ and $\underline{\rho}(D, x) \equiv \min_{\rho^* \in \Gamma} \rho^*(D, x)$, respectively. First, we rewrite Γ based on the network representation. By applying Lemma 1 to condition (β), we have

$$\Gamma = \left\{ \rho^* \in \mathbf{R}_+^{\mathcal{M}^*} \;\middle|\; \rho^* \text{ satisfies } (\alpha) \text{ and } \exists r \in \mathbf{R}_+^{\mathcal{M}^*} \text{ satisfying (5), (6), and (7)} \right\}.$$

By the Möbius inversion formula, (7) is equivalent to $\rho^*(D,x) = \sum_{E:E\supseteq D} r(E \setminus x, E)$ for all $(D,x) \in \mathcal{M}^*$, and moreover, (α) is rewritten as

$$\text{(25)} \qquad \sum_{E:E\supseteq D} r(E\setminus x, E) = \rho(D,x) \quad \text{for all } (D,x)\in\mathcal{M}.$$

Hence, we have

$$\Gamma = \left\{ \left(\sum_{E:E\supseteq D} r(E\backslash x, E) \right)_{(D,x)\in\mathcal{M}^*} \middle| \text{For some } r\in\mathbf{R}_+^{\mathcal{M}^*} \text{ satisfying (5), (6), and (25)} \right\}.$$

The Möbius inversion formula implies that (25) holds if and only if

$$\text{(26)} \qquad r(D\setminus x, D) = K(\rho, D, x) \quad \text{for all } (D,x)\in\mathcal{M}.$$

By substituting equalities (26) into conditions (5) and (6), it is shown that these two conditions are equivalent to (16). Thus, we obtain

$$\Gamma = \left\{ \left(\sum_{E:E\supseteq D} r(E\backslash x, E) \right)_{(D,x)\in\mathcal{M}^*} \middle| \text{For some } r\in\mathbf{R}_+^{\mathcal{M}^*} \text{ satisfying (26) and (16)} \right\}.$$

Now, for $(D,x)\in\mathcal{M}^*\backslash\mathcal{M}$, the upper bound of Γ is given by $\overline{\rho}(D,x) = \max_{\rho^*\in\Gamma}\rho^*(D,x) = \max_{r\in\mathbf{R}_+^{\mathcal{M}^*}\text{s.t. (26) and (16)}} \sum_{E:E\supseteq D} r(E\setminus x, E)$. The lower bound is obtained analogously.

E Summary of Liao, Saito, Sandroni (2025)

In this section we briefly summarize the main findings of the companion paper. In the present paper, we introduce an outside option x_0 only for choice sets that contain all unobservable alternatives X^*. For any nonempty set $A \subseteq \tilde{X} \equiv X\setminus X^*$, we define the reduced dataset $\hat{\rho}$ on menus $A\cup\{x_0\}$ by

$$\text{(27)} \qquad \hat{\rho}\big(A\cup\{x_0\}, x_0\big) \;\equiv\; 1 - \sum_{a\in A}\rho\big(A\cup X^*,\, a\big).$$

Note that (27) is equivalent to the definition of $\hat{\rho}$ in Definition 6. For example, if $X^* = \{c, d\}$, then $\hat{\rho}(A \cup x_0, x_0) = \rho(A \cup \{c, d\}, \{c, d\})$.

In Liao, Saito and Sandroni (2025), we consider a more general setting in which the composition of the outside option is described as an unknown distribution to the analyst. More precisely, for $A \subseteq \tilde{X}$ and $E \subseteq X^*$, we denote by $\lambda_{A\cup x_0}(E)$ the probability that the outside option x_0 represents the set E of unobservable alternatives. We call λ the *composition distribution*. We say that $\hat{\rho}$ is *RU-rationalizable* if there exists a composition distribution λ and a distribution μ over linear orders on the observable alternatives as well as the unobservable alternatives such that for all $a \in A$:

$$\hat{\rho}(A \cup x_0, a) = \sum_{E \in 2^{X^*} \setminus \{\emptyset\}} \lambda_{A\cup x_0}(E) \mu(\succ \mid a \succ b \text{ for all } b \in (A \cup E) \setminus \{a\}). \tag{28}$$

Note that the formula (28) implies that the probability that the outside option is chosen from $A \cup x_0$ equals

$$\hat{\rho}(A \cup x_0, x_0) = \sum_{E \in 2^{X^*} \setminus \{\emptyset\}} \lambda_{A\cup x_0}(E) \rho(A \cup E, E), \tag{29}$$

where $\rho(A \cup E, E) \equiv \sum_{x^* \in E} \rho(A \cup E, x^*)$. For instance, if $X^* = \{c, d\}$, then $\hat{\rho}(A \cup x_0, x_0) = \lambda_{A\cup x_0}(\{c, d\})\rho(A \cup \{c, d\}, \{c, d\}) + \lambda_{A\cup x_0}(\{c\})\rho(A \cup \{c\}, c) + \lambda_{A\cup x_0}(\{d\})\rho(A \cup \{d\}, d)$. Note also that when $\lambda_{A\cup x_0}(X^*) = 1$, this formula (29) coincides with (27).

One of the main results in the companion paper shows that $\hat{\rho}$ is RU-rationalizable if and only if: (i) $\hat{\rho}$ is RU-rationalizable on all choice sets that do not include the outside option; (ii) For every choice set E that does not contain x_0, and for all $y \in E$, we have $\hat{\rho}(E, y) \geq \hat{\rho}(E \cup x_0, y)$.

This result shows that the outside option approach imposes extremely weak constraints on the data. For the present paper, the importance of this result is that these two implications are *not* enough to guarantee the existence of a distribution $\hat{\mu}$ over linear orders on $\hat{X}$ that rationalizes the reduced dataset.

The companion paper also shows that the naive approach studied in the present paper, in which λ is constant across choice sets, serves as an idealized benchmark where the model retains relatively strong implications in the sense that it guarantees the existence of a distribution $\hat{\mu}$ that rationalizes the reduced dataset. As such, our current setup provides a natural starting point for analyzing the limitations of the outside option approach. In the present paper, we show that even under this idealized scenario, key implications of the RUM may still be lost.

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