Question

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Let X be the set of alternatives. Let $\mathcal{D} \subset 2^X \setminus \emptyset$ be the set of choice sets.

Stochastic choice function: A function $\rho : \mathcal{D} \times X \to [0,1]$ is called a *stochastic choice function* if $\sum_{x \in D} \rho(D,x) = 1$ and $\rho(D,x) = 0$ for any $x \notin D$.

For each $(D,x) \in \mathcal{D} \times X$, the number $\rho(D,x)$ is the probability that an alternative x is chosen from a choice set D.

Rankings: Let Π be the set of bijections between X and $\{1, \ldots, |X|\}$, where $|X|$ is the number of elements of X. If $\pi(x) = i$, then we interpret x to be the $|X| + 1 - i$-th best element of X with respect to π. If $\pi(x) > \pi(y)$, then x is better than y with respect to π. An element π of Π is called a *strict preference ranking* (or simply, a *ranking*) over X. For all $(D,x) \in \mathcal{D} \times X$ such that $x \in D$, if $\pi(x) > \pi(y)$ for all $y \in D \setminus \{x\}$, then we often write $\pi(x) \geq \pi(D)$. There are $|X|!$ elements in Π.

$$\rho^\pi(D,x) = \begin{cases} 1 & \text{if } \pi(x) \geq \pi(D); \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The function ρ^π gives probability one to the best alternative x in a choice set D according to the strict preference ranking π. The following fact is elementary but fundamental:

We denote the set of probability measures over Π by $\Delta(\Pi)$. Since Π is finite, $\Delta(\Pi) = \left\{(\nu_1, \ldots, \nu_{|\Pi|}) \in \mathbf{R}_+^{|\Pi|} \,\middle|\, \sum_{i=1}^{|\Pi|} \nu_i = 1\right\}$, where $\mathbf{R}_+$ is the set of nonnegative real numbers.

We now introduce the definition of random utility model:

Definition 1. A stochastic choice function ρ is called a *random utility function* if there exists a probability measure $\nu \in \Delta(\Pi)$ such that for all $(D, x) \in \mathcal{D} \times X$, if $x \in D$, then

$$\rho(D, x) = \nu(\pi \in \Pi | \pi(x) \geq \pi(D)).$$

The probability measure ν is said to *rationalize* ρ. The set of random utility functions is denoted by $\mathcal{P}_r$.[1]

Observation: *The set $\mathcal{P}_r$ of random utility functions is a polytope, that is $\mathcal{P}_r = co.\{\rho^\pi | \pi \in \Pi\}$.*

The observation holds because any random utility function $\rho \in \mathcal{P}_r$ can be written as $\rho = \sum_{\pi \in \Pi} \nu(\pi)\rho^\pi$ for some probability measure ν, which is guaranteed to exist by the definition of a random utility function.

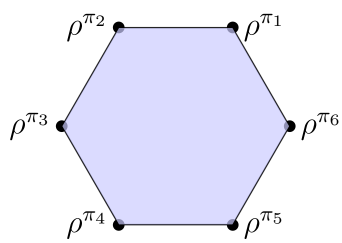

Figure 1: Random utility polytope

The hexagons in Figure 1 illustrates the polytope. Although the geometric intuition is useful, it is important to notice that the figure oversimplifies the reality since the number (i.e., $|X|!$) of vertices and the dimension of a random utility function can be very large.[2]

One concept that is useful for analyzing the polytope is *adjacency*. As you can see, in Figure 1, ρ^{π_i} and $\rho^{\pi_{i+1}}$ are *adjacent* for each $i \leq 6$.[3]

This can be formalized as follows:

Definition 2. *The two rankings π and π' are adjacent if there exists $t \in \mathbf{R}^{\mathcal{D} \times X}$ and $a \in \mathbf{R}$ such that (i) $\rho^\pi \cdot t = a = \rho^{\pi'} \cdot t$ and (ii) for any $\hat{\pi}$, if $\pi \neq \hat{\pi} \neq \pi'$, then $\rho^{\hat{\pi}} \cdot t > a$, where $\rho^\pi \cdot t \equiv \sum_{(D,x): x \in D \in \mathcal{D}} \rho^\pi(D, x) t(D, x)$ is the inner product.*

[1] While the function above is often called a random ranking function, a random utility function is often defined differently–by using the existence of a probability measure μ over utilities such that for all $(D, x) \in \mathcal{D} \times X$, if $x \in D$, then $\rho(D, x) = \mu(u \in \mathbf{R}^X | u(x) \geq u(D))$.

[2] To see why the dimension of a random utility function can be very large, notice that it assigns a number for each pair of $(D, x) \in \mathcal{D} \times X$.

[3] Assuming $\rho^{\pi_7} = \rho^{\pi_1}$.

For example, take ρ^{π_1} and ρ^{π_2}. Consider a line that passes through ρ^{π_1} and ρ^{π_2}. Notice that only ρ^{π_1} and ρ^{π_2} are on the line and all of the other $\{\rho^{\pi_i}\}_{i\neq 1,2}$ are below the line. Thus, conditions (i) and (ii) are indeed satisfied.

Question: Please answer the following two questions assuming $X = \{1, 2, 3\}$. :

Hint: In this case, there are six rankings: $(1, 2, 3), (1, 3, 2), ..., (3, 2, 1)$. For each ranking, you can define ρ^{π} in the space of $\mathcal{D} \times X$ following (1).

Question 1: Assume $\mathcal{D} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Find all adjacent vertices to the ranking $(1, 2, 3)$. Do your best to try to find all of them.

Question 2: Assume $\mathcal{D} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Find all adjacent vertices to the ranking $(1, 2, 3)$. Do your best to try to find all of them.

Question 3: Assume that $|X| \geq 4$. Fix $\pi, \pi' \in \Pi$. Suppose that π, π' are not adjacent. Show that there exist $\hat{\pi}, \hat{\pi}' \in \Pi$ such that

$$\frac{1}{2}\rho^{\pi} + \frac{1}{2}\rho^{\pi'} = \frac{1}{2}\rho^{\hat{\pi}} + \frac{1}{2}\rho^{\hat{\pi}'}.$$

(This equality means that we have the equality for each (D, x) such that $x \in D \in \mathcal{D}$.)