Random Utility with Unobservable Alternatives

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Abstract

The random utility model (RUM), a cornerstone in economics, has been studied under the assumption of observability of alternatives. In practice, it is common for choice frequencies of some alternatives to remain unobserved, for example data with an outside option. To address this discrepancy, we obtain the testable implications of the RUM when the choice frequencies of some alternatives are unobservable. These implications consist of nonredundant inequality constraints on observed choice frequencies. Our findings indicate that the practice of aggregating unobserved alternatives into a single outside option may be problematic as it fails to capture significant implications of RUM.

Keywords: Random utility, axiom, network flow, polytope

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1 Introduction

Consider a population of individuals choosing an alternative from various choice sets, where the analyst observes the choice frequencies of each alternative within each set. A foundational framework for interpreting such datasets is the random utility model (RUM), which posits a probability distribution over rankings of alternatives, with each ranking representing an individual's preferences. This model serves as a central tool in economics for linking observed stochastic choice behavior to underlying preference structures.

Falmagne (1978) axiomatized the RUM under the assumption that choice frequencies for all alternatives in all menus are observable. However, it is often the case that the choice frequencies for some alternatives are not observed. For example, consider a set of transportation methods consisting of bus, train, walking, and driving. While it may be possible to estimate the market share of public transportation (bus or train) based on the revenue of bus or train companies, it can be difficult to determine whether a person chooses to drive or walk unless a survey is conducted. As a result, the choice frequencies for walking and driving may not be available.

There are many other economically significant examples in which the choice frequencies of some alternatives are not observable. In such situations, empirical researchers often aggregate all unobservable alternatives into a single category and treat it as a generic *outside option*, even when they know which specific alternatives are available. We refer to this approach as *the outside option approach*. With this approach, all choice frequencies become observable because there is only one outside option, and the choice frequencies of all other alternatives are known. The choice frequency of the outside option is then calculated as one minus the sum of all observable choice frequencies.

The purpose of this paper is twofold. First, we investigate a necessary and sufficient condition for a RUM to rationalize the observed choice data when the choice frequencies of some alternatives are unobservable. Second, through this investigation, we demonstrate the limitations of the outside option approach. We show that by relying on the outside option approach, researchers may believe that a RUM applies when it does not but also overlook valuable information contained in the dataset. One key takeaway from these results is that empirical researchers should use the outside option approach with caution, making a deliberate effort to specify available alternatives in each choice set as much as possible. By doing so, the researchers can exploit all the implications of the RUM. In Section 1.1, we demonstrate these points by providing an example. In the following, we elaborate on these two objectives in order.

We focus on a setup where the choice frequencies of some alternatives are consistently missing, while those for the other alternatives are observable. For such datasets, we derive a finite system of linear inequalities that provides a necessary and sufficient condition for the dataset to be rationalized by a RUM. Furthermore, the characterization we provide is nonredundant, meaning that none of the inequalities are implied by any others, and removing even one of them would render the condition insufficient.[1] As we explain later, the nonredundancy of the conditions facilitates us to show the limitation of the outside option approach.

Our necessary and sufficient condition consists of two key components. The first is the classical nonnegativity of the Block-Marschak polynomials, a condition that appears in Falmagne's characterization. The second is a novel condition that consists of inequalities involving the summation and subtraction of the Block-Marschak polynomials. This novel condition highlights that the nonnegativity of the Block-Marschak polynomials alone is insufficient to rationalize a dataset. Instead, balances across the values of the Block-Marschak polynomials are crucial when dealing with incomplete datasets.

Our findings have significant implications for the widely used outside option approach. We show that the outside option approach disregards all but the most basic inequalities of the second condition. Importantly, this conclusion is made possible by the nonredundancy of our characterization. In particular, our results show that even if the dataset is not rationalizable by any RUM, the outside op-

[1] Falmagne (1978)'s characterization is also almost nonredundant. Suck (2002) and Fiorini (2004) show that omitting just a few inequalities from Falmagne (1978) results in a nonredundant characterization. The characterization by McFadden and Richter (1990) involves infinitely many inequalities and entails redundancy.

tion approach may erroneously conclude that the population of agents' choices is rationalizable by a RUM.

To establish our results, we translate the problem into a network flow problem. This approach was originally developed by Fiorini (2004), who provided a shorter proof of the axiomatization of Falmagne (1978). Our methodological innovation is employing a feasibility theorem in network flow theory, which provides a necessary and sufficient condition for the existence of a desirable network flow. This novel tool offers clear insights even in cases where some alternatives are missing. Moreover, we demonstrate that our methodology not only facilitates characterization but also significantly enhances efficiency when testing our conditions in given datasets.

1.1 Motivating Example

To demonstrate the importance of analyzing datasets without relying on the outside-option approach, we provide an example, in which the original dataset cannot be rationalized by any RUM, but under the outside-option approach, the dataset appears to conform to a RUM. Moreover, in the example, with the outside option approach, we lose critical information about the desirability of alternatives.

Let $\{a, b, c, d\}$ be the set of alternatives. Suppose that we do not observe choice frequencies of alternatives c and d. The left table in Table 1 shows the observable choice frequencies $\rho(D, x)$ of alternative x from D. The right table in Table 1 shows a reduced dataset $\hat{\rho}$ in which an outside option x_0 represents the set of all unobservable alternatives $\{c, d\}$. (See Definition 3.4 in Subsection 3.1 of the paper for the formal definition of the reduced dataset.)

Note first that in Table 1, when there is only one unobservable alternative in a choice set, we can calculate the frequency of the unobservable alternative by subtracting the sum of observable choice probabilities from one. For example, $\rho(\{a, c\}, c) = 1 - \rho(\{a, c\}, a) = 1/3$.

Note also that ρ cannot be rationalized by any RUM. One easy way to see this is to note that monotonicity is violated in ρ (i.e., $\rho(\{a, c\}, c) \not\geq \rho(\{a, b, c\}, c)$). Crucially however, this violation disappears in the reduced model $\hat{\rho}$ and in fact one can verify that $\hat{\rho}$ is rationalizable by a RUM, even though the original dataset

ρ

$D \backslash x$	a	b	c	d
$\{a,b\}$	$1/2$	$1/2$	$-$	$-$
$\{a,c\}$	$2/3$	$-$	$1/3$	$-$
$\{a,d\}$	$2/3+\varepsilon$	$-$	$-$	$1/3-\varepsilon$
$\{b,c\}$	$-$	$1/2$	$1/2$	$-$
$\{b,d\}$	$-$	$1/2+\varepsilon$	$-$	$1/2-\varepsilon$
$\{c,d\}$	$-$	$-$	?	?
$\{a,b,c\}$	$1/3$	$1/6$	$1/2$	$-$
$\{a,b,d\}$	$1/3+\varepsilon/2$	$1/6+\varepsilon/2$	$-$	$1/2-\varepsilon$
$\{a,c,d\}$	$1/3$	$-$	?	?
$\{b,c,d\}$	$-$	$1/3$	?	?
$\{a,b,c,d\}$	$1/6$	$1/6$	?	?

$\hat{\rho}$

$D \backslash x$	a	b	x_0
$\{a,b\}$	$1/2$	$1/2$	$-$
$\{a,x_0\}$	$1/3$	$-$	$2/3$
$\{b,x_0\}$	$-$	$1/3$	$2/3$
$\{a,b,x_0\}$	$1/6$	$1/6$	$2/3$

Table 1: The left shows the original choice probabilities $\rho(D,x)$ and the right shows the reduced choice probabilities $\hat{\rho}(D,x)$. The rows indicate choice set D, the columns indicate alternative x, and question marks indicate unobservable choice probabilities. We assume $0 \leq \varepsilon \leq 1/3$.

is not.[2]

Additionally, in the original dataset, $\rho(D,a) \geq \rho(D,b)$ for all D with $\{a,b\} \subseteq D$ and $\rho(D \cup a, a) \geq \rho(D \cup b, b)$ for any D such that $a \notin D$ and $b \notin D$. Moreover, these inequalities are strict when $D = \{c\}$ or $D = \{d\}$. Thus it can be inferred that a is strictly more desirable than b. However, in the reduced dataset, the desirability of a and b is indistinguishable. This illustrates how the estimated desirability of a and b may be biased when using the outside-option approach.

Another consequence of the outside option approach is that in the reduced dataset c and d are indistinguishable. In the original dataset, however, we can compare choice sets with different unobservable alternatives to learn about their relative desirability. In particular, we can learn that c is more desirable than d. For example, we have that $\rho(D \cup c, c) > \rho(D \cup d, d)$ for all non-empty D such that $c \notin D$ and $d \notin D$. We can also make inferences about the choice

[2] The reduced dataset is rationalizable by a distribution μ such that $\mu(\succ_1) = 1/3$, $\mu(\succ_2) = 1/3$, $\mu(\succ_3) = 1/6$, and $\mu(\succ_4) = 1/6$, where $x_0 \succ_1 a \succ_1 b$, $x_0 \succ_2 b \succ_2 a$, $a \succ_3 b \succ_3 x_0$, and $b \succ_4 a \succ_4 x_0$.

frequencies of c and d even when both are in the choice set. For example, suppose we only observe the choice sets $\{b,c,d\}$, $\{b,c\}$, and $\{b,d\}$. Assuming monotonicity of choice frequencies, we may conclude $\rho(\{b,c,d\},c) \leq \rho(\{b,c\},c) = 1/2$ and $\rho(\{b,c,d\},d) \leq \rho(\{b,d\},d) = 1/2 - \varepsilon$.[3] Then $\rho(\{b,c,d\},c) = 1 - \rho(\{b,c,d\},b) - \rho(\{b,c,d\},d) \geq 1 - 1/3 - (1/2 - \varepsilon) = 1/6 + \varepsilon$. Therefore when $\varepsilon > 1/6$, we have that $\rho(\{b,c,d\},d) \leq 1/2 - \varepsilon < 1/3 < 1/6 + \varepsilon \leq \rho(\{b,c,d\},c)$. We conclude that, under monotonicity (and therefore RUM) the probability that c is chosen from $\{b,c,d\}$ is higher than the probability that d is chosen from the same menu. All of these conclusions suggest that c is more desirable than d, an information that is lost in the reduced dataset.

In the example of Table 1, there are only two unobservable alternatives, so we can directly infer the choice frequency of each unobservable alternative when the choice set contains only one unobservable alternative. However, even when there are more than two unobservable alternatives in a choice set, we can still learn about the sum of their choice frequencies, which enables us to compare the desirability of different unobservable alternatives. In this sense, our conclusion that our methods can extract more information than the outside option approach does not depend on there being only two unobservable alternatives in the dataset. (See Section 4 for further details.)

This example highlights the importance of analyzing the original dataset without introducing an outside option. More specifically, by identifying which alternatives are available to the agent for each menu, we can infer more information about the agents' preferences. In the previous example, we showed that it is possible to learn more about the relative desirability between the observable alternatives a and b as well as the unobservable alternatives c and d simply by observing the choice probabilities on choice sets containing some but not all unobservable alternatives.

These observations underscore the limitations of the outside-option approach. In this paper, we formalize those limits by first characterizing the full implications of the RUM when some choice frequencies are unobservable (Theorem 3.2) and by identifying which implications are lost under the outside-option approach (Propo-

[3] Monotonicity means that $\rho(E,x) \leq \rho(D,x)$ for all $x \in D \subseteq E$.

sition 3.5). Moreover, in Section 4, we formalize our methods to obtain bounds for the unobserved choice frequencies. In particular, we provide an efficient algorithm that exploits the problem's network-flow structure to compute tight bounds, offering a practical tool for further analysis. Using real datasets, we show that our bounds are substantially tighter than the naïve bounds derived from the outside-option approach and that they recover information about the relative desirability of unobservable alternatives that the outside-option approach entirely discards (see Remark 4.2 and Proposition 4.4).

1.2 Related Literature

It is well known that obtaining a nonredundant characterization of the RUM with incomplete datasets in general is a challenging problem. See Martí and Reinelt (2011) for a survey. A more recent paper by Sprumont (2022) also highlights the difficulty of this problem. For example, when choice frequencies are observed only on binary choice sets, it has been unknown how to obtain a nonredundant characterization of the RUM since the 1980s; only for the case where the number of alternatives is less than eight, the nonredundant characterizations have been obtained.[4]

McFadden and Richter (1990) have proposed a characterization of the random utility models for incomplete datasets. Our result differs from their results in that their characterization involves infinitely many inequalities and entails redundancy, while ours consists of finite inequalities and contains no redundant ones.[5] A well-known version of McFadden and Richter (1990)'s axiom is not sufficient to characterize RUM in the presence of unobservable alternatives, although the original version of the axiom may be adapted to our setup. See Section G.3 of the

[4]See Reinelt (1993). This is in contrast with logit model: The logit models can be axiomatized with binary choice sets. See Luce (2005), Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2023), and Cerreia-Vioglio, Lindberg, Maccheroni, Marinacci, and Rustichini (2021) for the axiomatization of the logit models. Recently Petri (2023) has obtained an axiomatization of a special case of the RUM, *single-crossing RUMs* introduced by Apesteguia, Ballester, and Lu (2017) in a binary choice setup.

[5]More recently, Turansick (2023) provides an alternative axiomatization using the network flow approach.

online appendix for detail.

Other than the papers mentioned so far, only a few papers have studied the characterization of the RUM with incomplete data. McFadden (2006) considers a nested structure of choice sets: if choice frequencies are observable in a menu D, then choice frequencies are observable in any larger set E (i.e., $E \supseteq D$). In this paper, we also adopt this restriction of available choice sets. Suck (2016) addresses the truncated complete choice environment, in which only choice sets with at least $k \geq 2$ alternatives are observable. Nevertheless, to the best of our knowledge, our setup, in which the choice frequencies of some alternatives are missing, is novel in the literature. Moreover, these results are special cases of our theorem—cases in which there are no unobservable alternatives.

As mentioned, we use the network-flow theory to prove our results. Since the publication of Fiorini (2004), some more recent papers have used the network-flow theory to investigate different topics on RUM. Turansick (2022) and Chambers and Turansick (2024) study the identification of RUMs. Chambers, Masatlioglu, and Turansick (2021) provide a new model of random utility with more than one agent. Doignon and Saito (2022) characterize the adjacency of vertices and facets of a *mulitiple-choice polytope* (i.e., the set of RUMs). None of these papers study incomplete datasets.

Finally, we highlight several studies that develop empirical methods for testing RUMs. Kitamura and Stoye (2018) propose a nonparametric statistical test for RUMs, focusing on datasets where choices are made from budget sets—a setting distinct from ours. Dean, Ravindran, and Stoye (2022) extend this framework to examine choice overload. While their methods are applicable to a variety of choice datasets, they are computationally intensive due to the inefficiency of the model representation they rely on. They rely on the representation of the set of RUMs as the convex hull of vertices (V-representation), but the number of vertices is huge. In contrast, our characterization, which takes advantage of a specific structure of data incompleteness, provides a more computationally efficient foundation for testing based on facet-defining hyperplanes (H-representation) of the RUM. See section G.2 of the online appendix for a further explanation.

2 Model

Let X be a finite set of alternatives. Let $X^* \subseteq X$ be the set of *unobservable alternatives*. We assume that the choice frequencies of the elements of X^* are not observable (even if a choice set includes the alternatives). Let $\tilde{X} \coloneqq X \setminus X^*$ be the set of *observable alternatives*.

Let $\mathcal{D} \subseteq 2^X \setminus \emptyset$ be the set of choice sets. Unlike Falmagne (1978), we do not assume that $\mathcal{D} = 2^X \setminus \emptyset$. Note that $(\mathcal{D}, \subseteq)$ is a partially ordered set, where $\subseteq$ is the set inclusion. Like McFadden (2006), we assume that $\mathcal{D}$ is an *upper set* (i.e., $\mathcal{D}$ satisfies the following: $D \in \mathcal{D},\ E \supseteq D \Longrightarrow E \in \mathcal{D}$). To make our notation simple, let $\mathcal{M} \coloneqq \{(D, x) \in \mathcal{D} \times \tilde{X} \mid x \in D\}$

Note that for any (D, x), the choice frequency over (D, x) is observable if and only if $(D, x) \in \mathcal{M}$ (i.e., $x \in \tilde{X}$ and $D \in \mathcal{D}$).

Definition 2.1. *A nonnegative vector $\rho \in \mathbf{R}_+^{\mathcal{M}}$ is called an incomplete dataset if it satisfies the following conditions: for any $D \in \mathcal{D}$,*

(i) if $D \subset \tilde{X}$, then $\sum_{x \in D} \rho(D, x) = 1$; and

(ii) if $D \not\subseteq \tilde{X}$, then $\sum_{x \in D \cap \tilde{X}} \rho(D, x) \leq 1$.

When the context is clear, we will simply call ρ a *dataset* instead of an *incomplete dataset*. If ρ is an incomplete dataset, then ρ is not defined on $(D, x) \notin \mathcal{M}$. This does not mean that we cannot know anything about the choice frequencies of elements in X^*. We can compute the total choice frequency of all unobservable alternatives in D as $1 - \sum_{x \in D \cap \tilde{X}} \rho(D, x)$.

Definition 2.2. *A nonnegative vector $\rho^* \in \mathbf{R}_+^{\{(D,x) \mid x \in D \in 2^X\}}$ is called a complete dataset if, for any $D \subseteq X$, $\sum_{x \in D} \rho^*(D, x) = 1$.*

2.1 Examples

Example 1 (Transportation): An analyst is often able to estimate the market share of public transportation methods (i.e., bus or train) based on the revenues of bus or train companies. However, it is sometimes difficult for the analyst to know separately the percentages of people who drive or walk. In this case, $X = \{\text{walk, drive, bus, train}\}$, $\tilde{X} = \{\text{bus, train}\}$ and $X^* = \{\text{walk, drive}\}$. An example of the set of choice sets is $\mathcal{D} = \big\{\{w,b,t\},\{w,d,b\},\{w,d,t\},\{w,d,b,t\}\big\}$, where w, d, b, and t stand for *walk*, *drive, bus*, and *train*, respectively. This set $\mathcal{D}$ can be obtained from the assumption that depending on the location of homes, some transportation methods are not available.[6]

Example 2 (Market Shares of Private Companies): One definition of market share is the percentage of a company's total sales divided by the market's total sales. The market's total sales can be estimated by consumer surveys. However, private companies occasionally do not disclose their financial information, including their total sales; thus the market shares of private companies are sometimes unobservable. For example, suppose that there are four companies (i.e., $X = \{a, b, c, d\}$). If companies c and d are private companies, then we do not know their sales (i.e., $c, d \notin \tilde{X}$). Other companies $\{a, b\}$ are public and the information from these companies is disclosed. In addition, the availability of products may vary across stores, which would give a variation of choice sets (i.e., $\mathcal{D}$).

Example 3 (School Choice for Private Schools): Applicants submit their choices among public schools so the government knows the percentage of students choosing each public school. However, it might not have access to information on how many students choose each private school. For example, suppose that there are four schools (i.e., $X = \{a, b, c, d\}$). Among them, c and d are private schools for which we do not know the choice frequencies (i.e., $c, d \notin \tilde{X}$). The availability of schools may depend on the location of homes, which would give a variation of choice sets (i.e., $\mathcal{D}$).

[6] Assuming that the analyst believes that the distribution of preferences is independent of the location of homes, it would make sense to find a single distribution over ranking that describes the choice frequencies across $\mathcal{D}$.

2.2 Random-Utility Rationalization

Let $\mathcal{L}$ be the set of linear orders on X, i.e., binary relations that are irreflexive, asymmetric, transitive, and weakly complete (i.e., for any distinct elements $x, y \in X$, either $x \succ y$ or $y \succ x$).

Definition 2.3. *An incomplete dataset ρ is random-utility (RU) rationalizable if there exists $\mu \in \Delta(\mathcal{L})$ such that, for any $(D, x) \in \mathcal{M}$, $\rho(D, x) = \mu(\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x)$. We then say that μ rationalizes ρ.*

Definition 2.4. *Let $p \in \mathbf{R}^{\{(D,x)|x \in D \in 2^X\}}$. For any (D, x) such that $x \in D \subseteq X$, define $K(p, D, x) = \sum_{E:E \supseteq D}(-1)^{|E \setminus D|}p(E, x)$. $K(p, D, x)$ is called a Block-Marschak (BM) polynomial.*[7]

Note that, given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$, the BM polynomial $K(\rho, D, x)$ can be calculated if and only if $(D, x) \in \mathcal{M}$ (i.e., $x \in \tilde{X}$ and $D \in \mathcal{D}$). The next remark provides an interpretation of a BM polynomial given a complete dataset ρ^*:

Remark 2.5. *Assuming that a complete dataset ρ^* is RU-rationalizable, we can provide an interpretation of a BM polynomial. Suppose that there exists $\mu \in \Delta(\mathcal{L})$ such that, for any $x \in D \subseteq X$, $\rho^*(D, x) = \mu(\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x)$. By the Möbius inversion formula, it follows that*

$$K(\rho^*, D, x) = \mu(\succ \in \mathcal{L} | D^c \succ x \succ D \setminus x), \tag{1}$$

where $D^c \succ x$ means that $y \succ x$ for all $x \in D^c$ and $x \succ D \setminus x$ means that $x \succ y$ for all $x \in D \setminus x$.[8] *Thus, $K(\rho^*, D, x)$ can be interpreted as a measure of population of agents whose preference satisfies $D^c \succ x \succ D \setminus x$.*

[7] As we will explain below, the BM polynomial is crucial concept to characterize RUMs. The BM polynomial appears in other contexts. For example, Brady and Rehbeck (2016) observe that one of their axioms is equivalent to a multiplicative version of the BM polynomial.

[8] To see this notice that $\mu(\succ \in \mathcal{L} | D^c \succ x \succ D \setminus x) = \mu(\bigcup_{E \supseteq D}\{\succ \in \mathcal{L} | E^c \succ x \succ E \setminus x\}) = \sum_{E \supseteq D} \mu(\succ \in \mathcal{L} | E^c \succ x \succ E \setminus x)$. Thus, we have $\rho^*(D, x) = \sum_{E \supseteq D} \mu(\succ \in \mathcal{L} | E^c \succ x \succ E \setminus x)$. By applying the Möbius inversion to this equation, we obtain $\mu(\succ \in \mathcal{L} | D^c \succ x \succ D \setminus x) = K(\rho^*, D, x)$.

3 Main Results

To characterize the RU-rationalizability of incomplete data, we define the following collection of choice sets:

Definition 3.1. *A nonempty collection* $\mathcal{C}$ *of subsets of* X *is called a test collection if there exists a set* $A \subseteq \tilde{X}$ *of observable alternatives and a nonempty upper set* $\mathcal{E} \subseteq 2^{X^*}$ *of unobservable alternatives such that* $\mathcal{C} = \{A \cup E \mid E \in \mathcal{E}\}$.[9] *Moreover, the test collection is said to be essential if* $\emptyset \neq A \neq \tilde{X}$ *and* $\mathcal{E} \neq 2^{X^*}$.

That is, a test collection is a set of menus that contain the same subset of observable alternatives and is closed to the addition of unobservable alternatives. The following is our main theorem.

Theorem 3.2. *(a) An incomplete dataset* $\rho \in \mathbf{R}_+^{\mathcal{M}}$ *is RU-rationalizable if and only if the following two conditions hold:*

- *(i) for any* $(D, x) \in \mathcal{M}$ *such that* $1 < |D| < |X|$*, the polynomial* $K(\rho, D, x)$ *is nonnegative; and*
- *(ii) for any essential test collection* $\mathcal{C} \subseteq \mathcal{D}$*,*

$$\left(\sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D \cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F \cup y, y) \right) \geq 0. \tag{2}$$

(b) The inequality conditions in (i) and (ii) are independent: for any inequality condition in (i) or (ii), there exists an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$ *that violates the inequality but satisfies all the other conditions in (i) and (ii). (Note that by statement (i), such* ρ *is not RU-rationalizable.)*

Recall that for any (D, x), the BM polynomial $K(\rho, D, x)$ is computable based on the observable data if and only if $(D, x) \in \mathcal{M}$. Thus, condition (i) is testable.

[9] Recall the property of an upper set: if $D \in \mathcal{E}, D \subseteq E \Longrightarrow E \in \mathcal{E}$. In the example in which $X^* = \{d, e\}$, all upper sets in 2^{X^*} are $\emptyset$, $\{\{d, e\}\}$, $\{\{d, e\}, \{d\}\}$, $\{\{d, e\}, \{e\}\}$, $\{\{d, e\}, \{d\}, \{e\}\}$, and $\{\{d, e\}, \{d\}, \{e\}, \emptyset\}$. The complement $\mathcal{E}^c$ is a *lower set* (i.e., $\mathcal{E}^c$ satisfies the following: $E \in \mathcal{E}^c, D \subseteq E \implies D \in \mathcal{E}^c$). We use the concept of lower set in the proof.

Also, when $\mathcal{C}$ is a test collection, $D \in \mathcal{C}$ and $D \cup x \notin \mathcal{C}$ imply that $x \in \tilde{X}$.[10] Thus, the first term as well as the second term in condition (ii) can be calculated based on the available data; so the condition (ii) is also testable.

When all alternatives are observable (i.e., $\tilde{X} = X$, equivalently $X^* = \emptyset$), our theorem specializes to the classical result of Falmagne (1978): a complete dataset is RU-rationalizable if and only if all Block–Marschak (BM) polynomials are nonnegative. Accordingly, condition (i) in our theorem requires that every BM polynomial that is computable from the incomplete data be nonnegative.

Novel conditions appear in (ii), which means that the nonnegativity of the BM polynomials is insufficient for the dataset to be RU-rationalizable because balances across the values of BM polynomials are essential for RU-rationalizability when the dataset is incomplete. For instance, a single BM value that is too large can trigger a violation if it appears with a negative sign. In Remark 3.3 and Subsection 3.2, we provide a further explanation of condition (ii).

Statement (b) of the theorem says that each inequality condition in (i) or (ii) cannot be implied by the other inequality conditions; thus it can be interpreted individually. As we explain later in Proposition 3.5, this feature is crucial to understand which implication of RUMs may be lost in the outside option approach.

Remark 3.3. *Suppose that an incomplete dataset ρ is RU-rationalizable by μ. To understand the interpretation of condition (ii) in Theorem 3.2, fix a test collection $\mathcal{C}$ and assume that $\mathcal{C}$ is a singleton set containing a set D. Since $\mathcal{C}$ is a test collection, we have $X^* \subseteq D$. Then, the left hand side of (2) simplifies to*

$$\sum_{x^* \in X^*} \mu(\succ \in \mathcal{L} | D^c \succ x^* \succ D \setminus x^*). \tag{3}$$

Thus the interpretation of the condition (ii) for this case is the non-negativity of the measure on population of agents whose preference satisfies $D^c \succ x^* \succ D \setminus x^*$ for some $x^* \in X^*$.

[10] Since $\mathcal{C}$ is a test collection, $\mathcal{C} = \{A \cup E | E \in \mathcal{E}\}$ for some $A \subseteq \tilde{X}$ and an upper set $\mathcal{E} \subseteq 2^{X^*}$. If $x \in X^*$, then $D \in \mathcal{C}$ implies $D \cup x \in \mathcal{C}$ by the definition of test collections (especially by the fact that $\mathcal{E}$ is an upper set).

Note that there is no testable implication of the nonnegativity of $\mu\big(\succ\in\mathcal{L}\,\big|\,D^c\succ x^*\succ D\setminus\{x^*\}\big)$ *for each* $x^*\in X^*$*, because* $K(\rho,D,x^*)$ *cannot be computed—unlike in the complete-data case (Remark 2.5). However, we can still compute* $\sum_{x^*\in D\cap X^*}K(\rho,D,x^*)$*, which coincides with the left-hand side of (2) and admits the interpretation in terms of* μ *given in (3). For the general form of condition (2) and the proof, see Section G.1 of the online appendix.*

3.1 Implication for the outside option approach

In the outside option approach, we represent the set X^* of all unobservable alternatives as a single alternative x_0. Thus, the set of all alternatives in the outside option approach is defined as $\hat{X}=\tilde{X}\cup x_0$, where $\tilde{X}$ is the set of observable alternatives. Consequently, we consider the *reduced* choice sets, denoted by $\hat{\mathcal{D}}$, which aggregates all elements of X^* into the outside option x_0.[11]

Definition 3.4. *Let* $\hat{\rho}\in\mathbf{R}_+^{\{(D,x)|x\in D\in\hat{\mathcal{D}}\}}$ *be the reduced dataset on the reduced choice set defined as follows. For all* $\hat{D}\in\hat{\mathcal{D}}$ *then define* $D=(\hat{D}\cap\tilde{X})\cup X^*$ *if* $x_0\in\hat{D}$ *and* $D=\hat{D}$ *otherwise. We define the choice frequencies of observable alternatives remain the same, i.e.,* $\hat{\rho}(\hat{D},x)=\rho(D,x)$ *for all* $x\in D\cap\tilde{X}$*, where* $\rho\in\mathbf{R}_+^{\mathcal{M}}$ *is the original (non-reduced) incomplete dataset. Then, the choice frequency of the outside option can be defined as* $\hat{\rho}(\hat{D},x_0)=1-\sum_{x\in D\cap\tilde{X}}\rho(D,x)$*.*

See Table 1 in Subsection 1.1 for the example of $\hat{\rho}$. We say that the reduced dataset $\hat{\rho}\in\mathbf{R}_+^{\{(\hat{D},x)|x\in\hat{D}\in\hat{\mathcal{D}}\}}$ is *RU-rationalizable* if there exists a probability distribution $\hat{\mu}$ on the set $\hat{\mathcal{L}}$ of linear orders on $\hat{X}$ such that for all $(\hat{D},x)$ such that $x\in\hat{D}\in\mathcal{D}$, $\hat{\rho}(\hat{D},x)=\hat{\mu}(\hat{\succ}\in\hat{\mathcal{L}}\mid x\mathrel{\hat{\succ}}y$ for all $y\in\hat{D}\setminus x)$.

Proposition 3.5. *Let* $\rho\in\mathbf{R}_+^{\mathcal{M}}$ *be an incomplete dataset. Suppose that* ρ *satisfies*

- *condition (i) of Theorem 3.2 and;*
- *condition (ii) of Theorem 3.2 for any singleton essential test collection* $\mathcal{C}\subseteq\mathcal{D}$*.*

[11] We ignore the data on choice sets that contain only some (but not all) element(s) of X^*. Formally, $\hat{\mathcal{D}}\coloneqq\{D\in\mathcal{D}\mid D\cap X^*=\emptyset\}\cup\{(D\setminus X^*)\cup x_0\mid D\in\mathcal{D},D\supseteq X^*\}\subseteq 2^{\hat{X}}$.

Then the reduced dataset $\hat{\rho} \in \mathbf{R}_+^{\{(\hat{D},x)|x\in\hat{D}\in\hat{\mathcal{D}}\}}$ is RU-rationalizable.

For the RU–rationalizability of the reduced dataset $\hat{\rho}$, Proposition 3.5 requires condition (ii) only for *singleton* essential test collections (in addition to condition (i)), whereas for the RU-rationalizability of the original dataset ρ, Theorem 3.2 imposes condition (ii) for *all* essential test collections (in addition to condition (i)).

To see the implication of the gap between Proposition 3.5 and Theorem 3.2 more clearly, suppose the original incomplete dataset ρ violates condition (ii) of Theorem 3.2 for some *non-singleton* essential test collection, while satisfying all other conditions of Theorem 3.2. Such ρ exists because of the independence of each inequality as stated in statement (b) of Theorem 3.2. Statement (a) of Theorem 3.2 implies that ρ is not RU-rationalizable. Nevertheless, Proposition 3.5 implies that the reduced dataset $\hat{\rho}$ is RU-rationalizable, thus researchers may erroneously conclude that the true data-generating process follows the RUM. In this way, the reduced dataset $\hat{\rho}$ loses critical implications of the original dataset ρ. In particular, Proposition 3.5 shows that the outside option approach discards substantial implications of RUM—all but the most basic inequalities in condition (ii).

Remark 3.6. *To illustrate the implication of Proposition 3.5, remember the incomplete dataset ρ in Table 1 in Subsection 1.1. Let $\varepsilon = 0$.*

- *One can observe that ρ violates condition (ii) with $\mathcal{C} = \{\{a, c, d\}, \{a, c\}\}$:*

$$\left(\sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D \cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F \cup y, y) \right) = -\frac{1}{6}$$

- *Therefore, by Theorem 3.2, the original dataset ρ is not RU-rationalizable. However, since ρ satisfies condition (i) and condition (ii) when $\mathcal{C}$ is a singleton, Proposition 3.5 implies that the reduced dataset $\hat{\rho}$ is RU-rationalizable, despite the fact that the true dataset ρ is not.*

Remark 3.7. *Our definition assumes that the composition of the outside option is constant across choice sets—that is, x_0 always represents the same set X^* of*

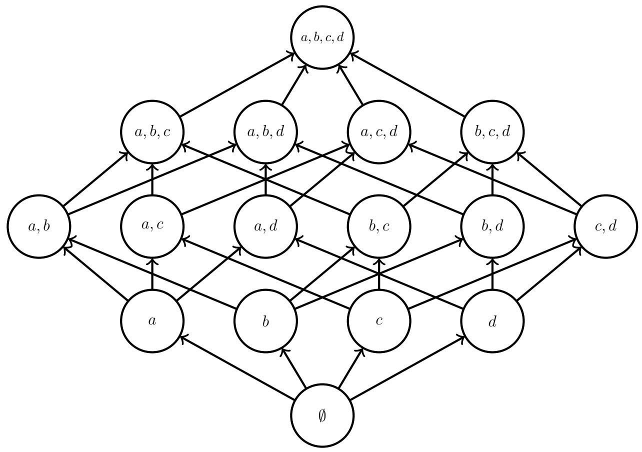

Figure 1: Network Flow for the case in which $X = \{a, b, c, d\} = \tilde{X}$ and $X^* = \emptyset$ and $\mathcal{D} = 2^X \setminus \emptyset$.

unobservable alternatives. Although this corresponds to the idealized setup often assumed in the empirical literature, it may not hold in real datasets: in practice, the interpretation of x_0 can vary across choice sets, and this variation is typically unobservable to the analyst. Liao, Saito, and Sandroni (2025) examines such cases by taking the reduced dataset as the primitive of the model. That paper shows that the implications of RUM are further weakened. See Section E in the appendix for more details.

3.2 Intuition of the proof

In this subsection, we outline the proof of statement (a) of Theorem 3.2. All formal proofs and the proof of statement (b) are in the appendix.

Before providing the sketch of our proof, we provide an overview of the network flow approach used by Fiorini (2004) for a complete dataset ρ^*. Recall that a *network* is a pair of a node set $\mathcal{N}$ and a set of directed arcs (i.e., edges) $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$. Two nodes s (source) and t (terminal) play special roles as explained below. We set $\mathcal{N} = 2^X, \mathcal{A} = \{(D, D \cup x) \mid D \subseteq X, x \notin D\}, s = \emptyset$, and $t = X$. See Figure 1 for an example. In the setup, each $\emptyset - X$ directed path corresponds to a unique ranking $\succ$. For example in Figure 1, the directed path $\emptyset - \{a\} - \{a, b\} - \{a, b, c\} - X$ corresponds to the ranking: $d \succ c \succ b \succ a$.

We now construct a vector (i.e., *flow*) $r \in \mathbf{R}_+^{\{(D,D\cup x)|x\in D\in 2^X\}}$ on the network assuming that a complete dataset ρ^* is RU-rationalizable by some $\mu \in \Delta(\mathcal{L})$, or

$$\rho^*(D,x) = \mu(\succ\in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x) \text{ for all } (D,x) \text{ s.t. } x \in D \subseteq X. \quad (4)$$

To each arc of the network, we assign the sum of the values of $\mu(\succ)$ over linear orders $\succ$ such that the directed path corresponding to $\succ$ goes through the arc. Given the construction, the value at an arc $(D\setminus x, D)$ is $\mu(\{\succ\in \mathcal{L} \mid D^c \succ x \succ D\setminus x\})$.[12] By the Möbius inversion formula, as explained in Remark 2.5, this value equals to $K(\rho^*, D, x)$. Note the constructed flow r satisfies all the following constraints:

$$\sum_{x\in X} r(X\setminus x, X) = 1, \quad (5)$$

$$\sum_{x\in D} r(D\setminus x, D) = \sum_{y\notin D} r(D, D\cup y) \text{ for any } D\in 2^X \text{ s.t. } 1\le |D| \le |X|-1, \quad (6)$$

$$r(D\setminus x, D) = K(\rho^*, D, x) \text{ for all } (D,x) \text{ such that } x\in D\in 2^X. \quad (7)$$

Condition (5) means that the sum of *inflows* to X (i.e. flows going into X) must be 1; the condition (6) means that for each node D, the sum of inflows to the node D equals to the sum of *outflows* from D (i.e. flows coming out from D); finally condition (7) means that the value of flow at an arc equals to the corresponding value of the BM polynomial.

The above observation shows that the three conditions are necessary for ρ^* to be RU-rationalizable. Fiorini (2004) proved that the conditions are also sufficient as follows:

Lemma 3.8. *Given a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$, there exists $\mu\in\Delta(\mathcal{L})$ satisfying (4) if and only if there exists $r \in \mathbf{R}_+^{\{(D\setminus x,D)|x\in D\in 2^X\}}$ satisfying (5), (6) and (7).*

[12] To see this, note that $\succ$ passes through the arc $(D\setminus x, D)$ if and only if x is ranked just above the elements of $D\setminus x$. That is, $\succ$ passes through the arc $(D\setminus x, D)$ if and only if $D^c \succ x \succ D\setminus x$. For example, the the path corresponding to $\succ$ passes through the arc $(\emptyset, \{a\})$ if and only if a is the worst element under $\succ$. Therefore the value assigned to the arc is equal to a measure over rankings whose worst elements are a.

It can be shown from the definition of the BM polynomial that (5) and (6) are satisfied automatically under the assumption of (7). Thus this implies the following:

Theorem (Falmagne (1978))**.** *Given a complete dataset* $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$*, there exists* $\mu \in \Delta(\mathcal{L})$ *satisfying (4) if and only if* $K(\rho^*, D, x) \geq 0$ *for all* (D, x) *such that* $x \in D \in 2^X$*.*

Now we will consider the case in which the dataset is incomplete. Remember in this case that $K(\rho, D, x)$ can be calculated if and only if $(D, x) \in \mathcal{M}$. We say an arc $(D \setminus x, D)$ is *observable* if $(D, x) \in \mathcal{M}$ (and hence we can calculate the value $K(\rho, D, x)$ of a flow at the arc); $(D \setminus x, D)$ is *unobservable* if $(D, x) \notin \mathcal{M}$ (and hence we *cannot* calculate the value $K(\rho, D, x)$ of a flow at the arc). See Figure 2, for illustration.

We extend the result by Fiorini (2004) to incorporate the case of incomplete datasets as follows:

Lemma 3.9. *Given an incomplete dataset* $\rho \in \mathbf{R}_+^{\mathcal{M}}$*,*
(P1) *there exists* $\mu \in \Delta(\mathcal{L})$ *such that*

$$\rho(D, x) = \mu(\succ \in \mathcal{L} \mid x \succ y \textit{ for all } y \in D \setminus x) \textit{ for any } (D, x) \in \mathcal{M} \tag{8}$$

if and only if
(P2) *there exists* $r \in \mathbf{R}_+^{\{(D\setminus x,D)|x\in D\in 2^X\}}$ *satisfying (5), (6), and*

$$r(D \setminus x, D) = K(\rho, D, x) \textit{ for all } (D, x) \in \mathcal{M}. \tag{9}$$

Note that unlike the case of complete data, Fiorini's approach (i.e., mapping the RU-rationalizability problem into a network flow) does not provide direct testable conditions without existential quantifiers.

In the following, we translate $(P2)$ to conditions without existential quantifiers. To provide an intuition on how to do so, consider Figure 3 in which $X = \{a, b, c, d\}$, $\tilde{X} = \{a, b\}$, $X^* = \{c, d\}$, and $\mathcal{D} = 2^X \backslash \emptyset$. Let $\mathcal{C} = \{\{a, c\}, \{a, d\}, \{a, c, d\}\}$.

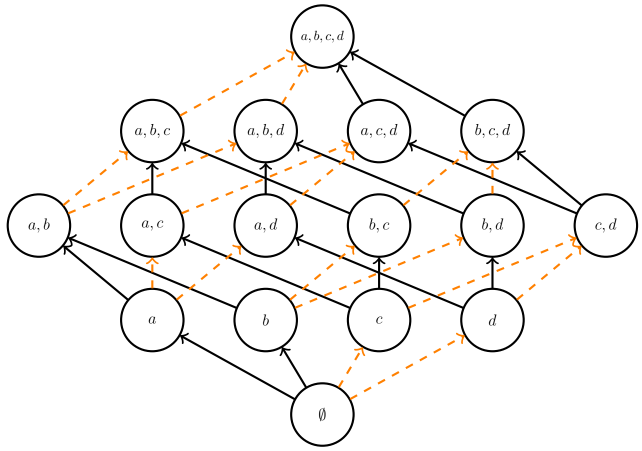

Figure 2: Network Flow for the case in which $X = \{a, b, c, d\}$, $\tilde{X} = \{a, b\}$, and $X^* = \{c, d\}$ and $\mathcal{D} = 2^X \setminus \emptyset$. The solid arrows correspond to observable arcs: the dotted arrows correspond to unobservable arcs.

In the figure, red flows are observable outflows from $\mathcal{C}$; yellow flows are unobservable inflows to $\mathcal{C}$; blue flows are observable inflows to $\mathcal{C}$. Note that there are no unobservable outflows.

By the equality between inflows and outflows with respect to $\mathcal{C}$, we have that the sum of red outflows equals to the sum of yellow inflows and the blue inflows. Although each yellow inflow is unobservable, we can calculate the sum of yellow inflows as the sum of red outflows minus the sum of blue inflows. If ρ is RU-rationalizable then the sum of yellow inflows is nonnegative, and hence *the sum of red outflows minus the sum of blue inflows must be nonnegative.* This is an implication that is *directly testable* based on the observable dataset since blue inflows and red outflows are observable. In fact, the sum of red outflows and the sum of blue inflows can be calculated as follows: (Red outflows)$= \sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D\cup x, x)$; (Blue inflows)$= \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F\cup y, y)$. Thus, the testable implication can be written as follows

$$\sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D\cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, E\cup y, y) \geq 0,$$

which means that *the net observable outflows from $\mathcal{C}$ must be nonnegative; and this*

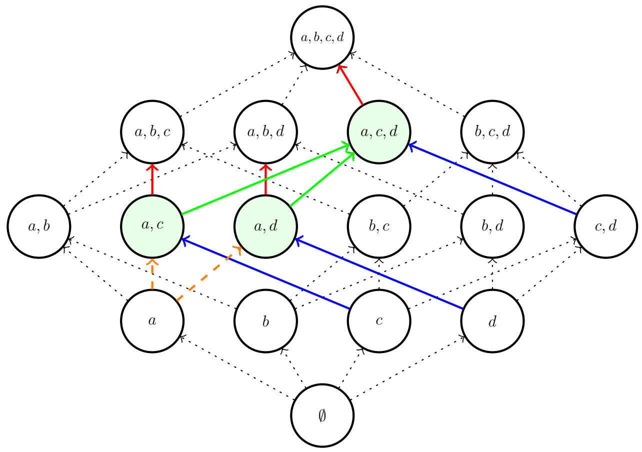

Figure 3: Outflows from $\mathcal{C} \equiv \{\{a,c\},\{a,d\},\{a,c,d\}\}$ and inflows to $\mathcal{C}$.

Note: Red flows are observable outflows from $\mathcal{C}$; yellow flows are unobservable inflows to $\mathcal{C}$; blue flows are observable inflows to $\mathcal{C}$. Note that green flows are flows contained in $\mathcal{C}$ and are not relevant to the net observable outflow from $\mathcal{C}$.

is the exactly one of the inequalities appearing condition (ii) in Theorem 3.2.

4 Bounds for Unobservable Choice Probabilities

In this section we obtain bounds for unobservable choice probabilities. In practice, predicting unobservable choice probabilities is important. Recall, for instance, the transportation example (Example 1) in Section 2.1. In this example, how people commute is not observable unless they use public transportation. That is, $X = \{\text{bus, train, walk, drive}\}$, $\tilde{X} = \{\text{bus, train}\}$, and $X^* = \{\text{walk, drive}\}$. Suppose that the government is considering introducing a new tax on gasoline to encourage people to commute by public transportation. To assess the potential impact of the new policy, it is crucial for the government to know the percentage of people who commute by private cars.

The most naive approach is merely to bound the fraction below by zero and above by the percentage of people who did *not* use public transportation. In Remark 4.1, we will observe that this naive approach corresponds to the outside option approach.

Remark 4.1. *Let $\rho \in \mathbf{R}_+^{\mathcal{M}}$ be a given incomplete dataset. Let $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in\hat{\mathcal{D}}\}}$*

be the reduced dataset in the outside option approach defined in Definition 3.4. Choose any (D, x^*) *such that* $D \in \mathcal{D}$, $X^* \subseteq D$, *and* $x^* \in X^*$. *The bounds for the unobservable choice frequency* x^* *from* D *derived from the restriction that* $\hat{\rho}$ *is RU-rationalizable are*

$$\Big[0, 1 - \sum_{a \in D \cap \tilde{X}} \rho(D, a)\Big]. \tag{10}$$

To see this, notice that every menu in the reduced dataset $\hat{\mathcal{D}}$ *either contains all or none of the unobservable alternatives. Therefore, there is no way to distinguish any unobservable alternative from another. Thus, with the outside option approach, RU-rationalizability does not have any implications beyond the fact that the probabilities must sum up to* 1.

4.1 Bounds derived from Monotonicity

In this section and the next, we show how the naïve bounds can be greatly tightened by exploiting implications of RUM. Here we focus on the simplest one–the monotonicity condition: for all $D, E \in \mathcal{D}$ and $x \in D$, if $D \subseteq E$ then $\rho(D, x) \geq \rho(E, x)$. This provides an intuitive explanation of how specifying, for each choice set, the available alternatives as precisely as possible enables sharper bounds.

Fix $D \in \mathcal{D}$ that contains some unobservable alternative $x^* \in X^*$. We will obtain the bound for the choice frequency of x^* from choice set D. The basic idea is that by comparing the choice probabilities on the choice sets D and $D \setminus y^*$ for each $y^* \in D \cap X^*$, we can learn about the relative desirability of each alternative $y^* \in D \cap X^*$.

Remark 4.2. *Let* ρ^* *be a complete dataset that coincides with* ρ *on* $\mathcal{M}$ *and satisfies monotonicity. Define* $F \equiv \{y^* \in D \cap X^* | D \setminus y^* \in \mathcal{D}\}$. *For any* (D, x^*) *such that* $x^* \in D \cap X^*$ *and* $D \in \mathcal{D}$, *the bounds for the unobservable choice frequency* $\rho^*(D, x^*)$ *become*

$$\Big[L(x^*), 1 - \sum_{a \in D \cap \tilde{X}} \rho(D, a) - \sum_{y^* \in (D \cap X^*) \setminus x^*} L(y^*)\Big], \tag{11}$$

where

$$L(y^*) = \begin{cases} \sum_{a\in D\cap\tilde{X}} \big(\rho(D\setminus y^*, a) - \rho(D,a)\big) & \textit{for all } y^* \in F, \\ 0 & \textit{otherwise.}^{13} \end{cases} \tag{12}$$

When $y^* \in F$, the lower bound $L(y^*)$ can be interpreted as the total substitution toward y^*, since each term $\rho(D\setminus y^*, a) - \rho(D,a)$ in the summation captures the substitution from alternative a to y^*. These bounds (11) allow us to compare the relative desirability of unobservable alternatives.[14]

By comparing the bounds (11) and the bounds (10) from the outside option approach obtained in Remark 4.1, the improvement of the bounds can be simply summarized as

$$\sum_{y^*\in D\cap X^*} L(y^*).$$

This quantity can be interpreted as the total substitution towards unobservable alternatives in D.

To see how to obtain the result, first notice that the upper bounds can be obtained easily given the lower bounds.[15] To get the lower bounds assume that $x^* \in F$. Then, by monotonicity, we have $\rho^*(D, y^*) \leq \rho^*(D\setminus x^*, y^*)$ for all $y^* \in X^*\setminus x^*$. Thus we have $\rho^*(D,x^*)+\sum_{a\in D\cap\tilde{X}} \rho(D,a) = 1-\sum_{y^*\in(D\cap X^*)\setminus x^*} \rho^*(D,y^*) \geq 1-\sum_{y^*\in(D\cap X^*)\setminus x^*} \rho^*(D\setminus x^*,y^*) = \sum_{a\in D\cap\tilde{X}} \rho(D\setminus x^*,a)$. It follows that $\rho^*(D,x^*) \geq \sum_{a\in D\cap\tilde{X}} \Big[\rho(D\setminus x^*,a) - \rho(D,a)\Big] \equiv L(x^*)$.

In the next remark, we demonstrate how to calculate the bounds by using example in Table 1 in Section 1.1

[13] Note that if ρ is not RU-rationalizable, it is possible that $L(x^*) > 1 - \sum_{a\in D\cap\tilde{X}} \rho(D,a) - \sum_{y^*\in(D\cap X^*)\setminus x^*} L(y^*)$ which we may interpret as an empty set.

[14] When $\mathcal{D}$ only contains menus of the form D and $D\setminus y*$ for $y^* \in X^*$ for some $D \subseteq X$, these bounds are actually tight. That is, they coincide with the RUM bounds derived in the next section. However, when the choice frequencies for other menus are available, monotonicity and RUM have further implications.

[15] Suppose that we obtain the lower bound $L(y^*)$ for all $y^* \in D\cap X^*$. To get the upper bound, simply subtract the lower bounds from 1 to obtain $\rho^*(D,y^*) \leq 1 - \sum_{a\in D\cap\tilde{X}} \rho(D,a) - \sum_{y^*\in D\cap(X^*\setminus x^*)} L(y^*)$.

Remark 4.3. *We will obtain bounds of unobservable choice frequencies from the choice set* $\{b, c, d\}$. *In the dataset* ρ *in Table 1, we observe* $\rho(\{b, d\}, b) = 1/2 + \varepsilon, \rho(\{b, c\}, b) = 1/2$, *and* $\rho(\{b, c, d\}, b) = 1/3$ *as alternative* b *is observable. From these we can calculate the lower bounds* $L(c)$ *and* $L(d)$ *as follows:*

$$L(c) = \rho(\{b, d\}, b) - \rho(\{b, c, d\}, b) = 1/2 + \varepsilon - 1/3 = 1/6 + \varepsilon,$$
$$L(d) = \rho(\{b, c\}, b) - \rho(\{b, c, d\}, b) = 1/2 - 1/3 = 1/6.$$

Thus, we also can obtain upper bounds for alternative c *and* d *as* $1-\rho(\{b, c, d\}, b)- L(d) = 1/2$ *and* $1 - \rho(\{b, c, d\}, b) - L(d) = 1/2 - \varepsilon$, *respectively. Thus, as observed in the introduction, the bounds for alternative* c *are* $[1/6 + \varepsilon, 1/2]$; *the bounds for alternative* d *are* $[1/6, 1/2 - \varepsilon]$.

When $\varepsilon \geq 1/6$, *observe that the bounds for alternative* c *locate higher than the bounds for alternative* d, *which suggests that alternative* c *is more desirable to alternative* d. *Thus Remark 4.2 generalizes the analysis done in Section 1.1.*

4.2 Bounds derived from RU rationality

In this section, we further improve the bounds by assuming full RU rationality based on our characterization.

Proposition 4.4. *Suppose that* ρ *is RU-rationalizable. Suppose also that* ρ^* *is a complete dataset that is RU rationalizable and coincides with* ρ *on* $\mathcal{M}$. *For any* $(D, x) \notin \mathcal{M}$, *the bounds for the unobservable choice frequency* $\rho^*(D, x)$ *are obtained by* $[\underline{\rho}(D, x), \overline{\rho}(D, x)]$, *where*

$$\overline{\rho}(D, x) = \max_{r \in \mathbf{R}_+^{\{(D,x)|x \in D \in 2^X\}}} \sum_{E:E \supseteq D} r(E \setminus x, E) \tag{13}$$

and

$$\underline{\rho}(D, x) = \min_{r \in \mathbf{R}_+^{\{(D,x)|x \in D \in 2^X\}}} \sum_{E:E \supseteq D} r(E \setminus x, E) \tag{14}$$

subject to the following constraint: for all nonempty $D \subseteq X$,

$$\sum_{(D,y)\notin\mathcal{M}:y\in D} r(D\setminus y, D) - \sum_{(D\cup y,y)\notin\mathcal{M}:y\notin D} r(D, D\cup y)$$
$$= 1\{D = X\} + \sum_{(D\cup y,y)\in\mathcal{M},y\notin D} K(\rho, D\cup y, y) - \sum_{(D,y)\in\mathcal{M},y\in D} K(\rho, E\cup y, y), \tag{15}$$

where $r(E \setminus x, E) = K(\rho, E, x)$ for all $(E, x) \in \mathcal{M}$.

Compared with obtaining the bounds directly from (P1), our bounds in Proposition 4.4 is computationally more efficient. This is because this problem can be seen as a minimum-cost transshipment problem, which is well known in the network-flow theory literature. One of the key properties of this problem is that it is a linear program with a constraint that has *an incidence matrix* as its coefficient.[16] For this specific problem, a practical polynomial time algorithm, called the *network simplex algorithm*, can be applied. (We refer readers to Orlin, Plotkin, and Tardos (1993) and Orlin (1997) for further computational aspects of the algorithm.) When $\mathcal{D} = 2^X \setminus \emptyset$, the bound (14) can be further simplified into (39) as shown in Section G.5 in the online appendix.

Kitamura and Stoye (2019) also consider bounds of counterfactual choice probabilities based on the formulation of Kitamura and Stoye (2018). Theoretically, their bounds coincide with Proposition 4.4, but it is possible that the bounds are much harder to compute than our bounds. This is because they compute the bounds directly from (P1), ignoring the network structure. Since the network simplex algorithm relies heavily on the fact that the constraint of the linear program is written with an incidence matrix, the original form (P1) does not have its benefit in terms of computational efficiency.

[16] An incidence matrix is a matrix representation of network structure, which is defined as a matrix consisting only of $0, 1$ and -1 with each column having exactly one element of 1 and -1.

4.3 Application to Lottery Data

To test our methods, we take a complete dataset from the experiment conducted by McCausland, Davis-Stober, Marley, Park, and Brown (2020) with all choice frequencies observable, then delete some of the choice frequencies to obtain an incomplete dataset. We apply our methods and then compare the bounds to the original choice frequencies. We find that that bounds obtained from our methods greatly improve on those from the outside option approach.

In the experiment, the authors fixed a set $X = \{0, 1, 2, 3, 4\}$ of five lotteries and asked 141 participants to choose one from each subset of X. Each participant made decision six times for each choice set. We aggregate these choice frequencies to construct a complete dataset denoted by ρ. The lottery dataset is nearly RU-rationalizable but not exactly. As our method can be applied only to RU-rationalizable datasets, we first fit a RUM to the dataset to get a calibrated dataset that is close to the original one and is RU-rationalizable. The detail of this procedure is in Subsection G.8 of the online appendix. We mask the choice probabilities of lotteries 0 and 1 and pretend not to observe them; in other words, we set $X^* = \{0, 1\}$, $\tilde{X} = \{2, 3, 4\}$, and $\mathcal{D} = 2^X \setminus \emptyset$.

Under this setup, we will compute three types of bounds on the probabilities of lotteries 0 and 1 being chosen in a given choice set D that contains both lotteries. The first type of bounds is (10) based on the outside option approach: $\left[0, 1 - \sum_{x \in D \cap \{2,3,4\}} \rho(D, x)\right]$. Note that the bounds for lotteries 0 and 1 are identical, as the outside option approach provides no information to distinguish among unobservable alternatives. The bounds are shown in blue dotted interval in Figure 4. We call these bounds *naive bounds.*

The second type of bounds is derived from the monotonicity condition obtained in Remark 4.2. As demonstrated in Remark 4.3, the bounds can be calculated easily given the dataset ρ.

The third and the last type of bounds is the one that exploits full implication of RUM and is computed by the linear program (14). (Since $\mathcal{D} = 2^X \setminus \emptyset$, we use (39) to calculate the values.) These bounds are shown in red for alternative 0 and in green for alternative 1 in Figure 4. We call these bounds *RUM bounds.*

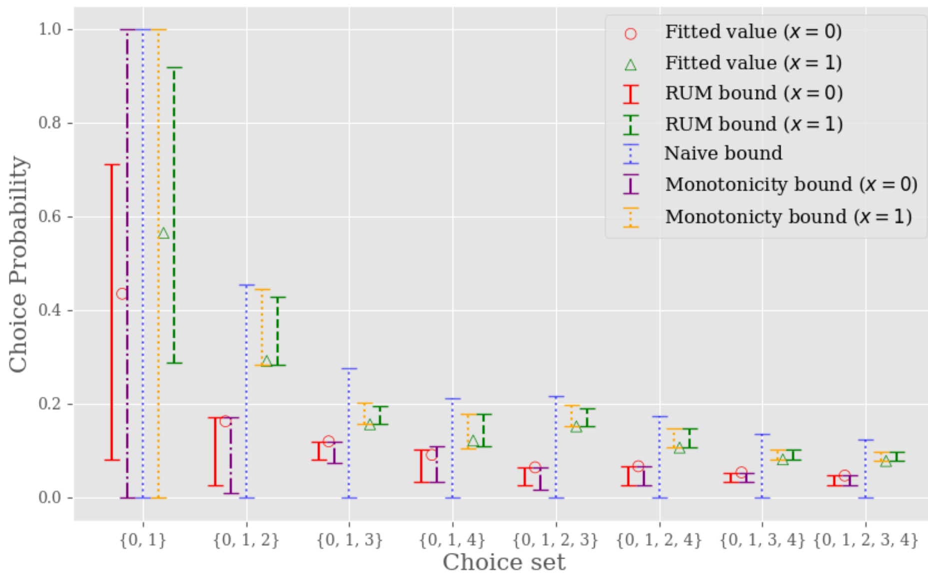

Figure 4: Comparison between the bounds of the choice probabilities of lotteries 0 and 1. The RUM bounds for 0 and 1 are shown in red and green, respectively. The monotonicity bounds for 0 and 1 are show in purple and yellow, respectively. The naive bounds for both are shown in blue.

There are two key takeaways from the figure. First, our bounds (i.e., the RUM bounds and the monotonicity bounds) are substantially tighter than the native bounds, particularly when the choice set is large. While the RUM bounds are slightly narrower than the monotonicity bounds, the two are quite similar except for the choice set $\{0, 1\}$.[17] This suggests that our methods effectively leverage additional information about unobserved choice frequencies. In particular, for the choice set $\{0, 1, 2, 3, 4\}$, the RUM bounds and the monotonicity bounds are extremely tight and closely approximate the true values.

Second, and more importantly, in the RUM bounds for the choice frequency

[17]While the given monotonicity bounds for $\{0, 1\}$ are not close to the RUM bounds, this is only because the bounds derived in Remark 4.2 use monotonicity in a simple way to aid the exposition. Indeed as mentioned in footnote 14, since there are additional menus beyond $\{0, 1\}$, $\{0\}$, and $\{1\}$, monotonicity has further implications beyond the given bounds. If these are taken into account, the monotonicity bound becomes very close to the RUM bound even on $\{0, 1\}$.

of the alternative 1 are always higher than those of alternative 0. Therefore, we can conclude that in any RU-rationalization, the probability that alternative 1 is chosen is higher than the probability that alternative 0 is chosen in all menus but $D = \{0, 1\}$. A similar conclusion can be made for the monotonicity bounds. On the other hand, in the outside option approach, the bounds for alternatives 1 and 0 are exactly identical. This difference indicates that our method exploits the information that alternative 1 is better than alternative 0, while the outside option approach loses this information completely. This observation suggests that the estimation of the desirability of unobserved alternatives would be biased in the outside option approach, though it is beyond the scope of the current paper.

5 Concluding Remark

In this paper, we consider a model of stochastic choice where some choice frequencies are unobservable. We formally demonstrate the limitations of the prevalent approach to analyzing such datasets in empirical research, the outside option approach. We begin by characterizing the full implications of the RUM when some choice frequencies are unobservable (Theorem 3.2), and then identify which implications are lost under the outside option approach (Proposition 3.5). Furthermore, in Remark 4.2 and Proposition 4.4, we develop methods to bound the unobserved choice frequencies, revealing information about the desirability of unobservable alternatives. This information is entirely lost in the outside option approach, as all unobservable alternatives are aggregated into a single alternative in that approach. Taken together, these results underscore the value of accurately identifying the set of alternatives available to each agent. In particular, our results suggest that accurate measurement of availability of each outside option enables a more reliable inference of consumer preferences.

Appendix

A Proof of Lemma 3.9

We first prove that (P1) implies (P2). Fix a solution μ to (P1). Define a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x) | x \in D \in 2^X\}}$ by $\rho^*(D, x) = \mu(\{\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x\})$

for all $x \in D \in 2^X$. Then $\rho^* = \rho$ on $\mathcal{M}$. Moreover, by Lemma 3.8, we have $r \in \mathbf{R}_+^{\{(D\setminus x, D) | x \in D \in 2^X\}}$ that satisfies (5), (6) and $r(D \setminus x, D) = K(\rho^*, D, x)$ for all (D, x) such that $x \in D \in 2^X$. Since $\rho^* = \rho$ on $\mathcal{M}$, thus we have $r(D \setminus x, D) = K(\rho, D, x)$ for all $(D, x) \in \mathcal{M}$. Thus, r is a solution to (P2).

Next we prove that (P2) implies (P1). Fix a solution r to (P2). For any $(D, x) \notin \mathcal{M}$, $\rho^*(D, x) \equiv \sum_{E: E \supseteq D} r(E \setminus x, E)$. Then $\rho^* = \rho$ on $\mathcal{M}$, where ρ is the given incomplete dataset. Thus, we obtain a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x) | x \in D \in 2^X\}}$. Then by the Möbius inversion, we have $r(D \setminus x, D) = K(\rho^*, D, x)$ for all (D, x) such that $x \in D \in 2^X$. Then r satisfies (5), (6), and (7). Then by Lemma 3.8, there exists $\mu \in \Delta(\mathcal{L})$ such that $\rho^*(D, x) = \mu(\{\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x\})$ for all (D, x) such that $x \in D \in 2^X$. Since $\rho = \rho^*$ on $\mathcal{M}$, (4) holds for any $(D, x) \in \mathcal{M}$. Thus, μ is a solution to (P1).

B Proof of Statement (a) of Theorem 3.2

To formalize the intuition explained in Section 3.2, we introduce a definition:

Definition B.1. *A collection $\mathcal{C} \subseteq 2^X$ is said to be complete if $D \in \mathcal{C} \implies \forall x \in X^*, D \cup x \in \mathcal{C}$.*

Note that a collection $\mathcal{C}$ of subsets is complete if and only if there are no unobservable outflows from $\mathcal{C}$.

We need one more definition: for any $\mathcal{C} \subseteq 2^X$, define

$$\delta_\rho(\mathcal{C}) = \left(\sum_{\substack{(D,x): D \in \mathcal{C}, D \cup x \notin \mathcal{C}, \\ (D \cup x, x) \in \mathcal{M}}} K(\rho, D \cup x, x) - \sum_{\substack{(E,y): E \notin \mathcal{C}, E \cup y \in \mathcal{C}, \\ (E \cup y, y) \in \mathcal{M}}} K(\rho, E \cup y, y) \right) + 1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}. \quad (16)$$

For any $\mathcal{C} \subseteq 2^X$, $\delta_\rho(\mathcal{C})$ is the net observable outflows from $\mathcal{C}$. To see this notice that the first term is the values of the observable outflows from $\mathcal{C}$ and the second term is the values of the observable inflows to $\mathcal{C}$.[18] δ_ρ satisfies the disjoint additivity as follows:

[18] The function δ can also be defined with $\mathcal{D} = 2^X \setminus \emptyset$ and $X^* \neq \emptyset$ as follows: for any complete

Lemma B.2. *For any* $\mathcal{C} \subseteq 2^X$, $\delta_\rho(\mathcal{C}) = \sum_{D\in\mathcal{C}} \delta_\rho(D)$.

Based on the discussion in Section 3.2, if a solution to $(P2)$ exists, then $\delta_\rho(\mathcal{C}) \geq 0$ for any complete collection $\mathcal{C}$. The next shows the converse also holds:

Lemma B.3. *Given an incomplete dataset* $\rho \in \mathbf{R}_+^{\mathcal{M}}$*, a solution to (P2) exists if and only if* $\delta_\rho(\mathcal{C}) \geq 0$ *for any complete collection* $\mathcal{C}$ *such that* $\emptyset \notin \mathcal{C}$.

Although Lemma B.3 together with Lemma 3.9 successfully characterizes RU-rationalizability based only on the available data, the condition has some redundancy. The next lemma provides a sharper characterization, which proves statement (a) of Theorem 3.2.[19]

Lemma B.4. *Given an incomplete dataset* $\rho \in \mathbf{R}_+^{\mathcal{M}}$*, a solution to (P2) exists if and only if (i)* $K(\rho, D, x) \geq 0$ *for all* $(D, x) \in \mathcal{M}$ *such that* $1 < |D| < |X|$ *and (ii)* $\delta_\rho(\mathcal{C}) \geq 0$ *for any essential test collection* $\mathcal{C} \subseteq \mathcal{D}$.

B.1 Proof of Lemma B.2

It suffices to show $\delta_\rho(\mathcal{C} \cup \mathcal{E}) = \delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$ for any disjoint sets $\mathcal{C}, \mathcal{E} \subseteq 2^X$. If there are no arcs connecting $\mathcal{C}$ and $\mathcal{E}$, then the result is trivial. Suppose otherwise. The values of a flow on the connecting arcs will be canceled out in $\delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$. Without loss of generality, suppose that there is a connecting arc $(D, D \cup x)$ and $D \in \mathcal{C}$ and $D \cup x \in \mathcal{E}$. Then the value $K(\rho, D \cup x, x)$ of the flow on the arc is added to $\delta_\rho(\mathcal{C})$ and subtracted from $\delta_\rho(\mathcal{E})$. Thus the value is canceled out in $\delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$. In the same way, the value $K(\rho, D \cup x, x)$ of the flow on the connecting arc $(D, D \cup x)$ and $D \in \mathcal{E}$ and $D \cup x \in \mathcal{C}$ will be canceled out.

B.2 Proof of Lemma B.3

To prove Lemma B.3, we first provide a general lemma. Fix a network $(\mathcal{N}, \mathcal{A})$. Remember that $\mathcal{N}$ is the set of nodes; $\mathcal{A}$ is the set of arcs. Consider a function

dataset ρ^*, define $\delta_{\rho^*}(\mathcal{C}) = \left(\sum_{\substack{(D,x)\\ D\in\mathcal{C}, D\cup x\notin\mathcal{C}}} K(\rho^*, D\cup x, x) - \sum_{\substack{(E,y)\\ E\notin\mathcal{C}, E\cup y\in\mathcal{C}}} K(\rho^*, E\cup y, y)\right)$ $+ 1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}$. We will use this definition later.

[19] We prove that the characterization is indeed nonredundant in Section C.

$r : \mathcal{A} \to \mathbf{R}$. For any node $D \in \mathcal{N}$, let $r(D, \mathcal{N}) \equiv \sum_{E \in \mathcal{N}} r(D, E)$; $r(\mathcal{N}, D) \equiv \sum_{E \in \mathcal{N}} r(E, D)$[20]. A function $r : \mathcal{A} \to \mathbf{R}$ is called a *flow* on a network $(\mathcal{N}, \mathcal{A})$ if it satisfies the following conditions:

$$r(s, \mathcal{N}) - r(\mathcal{N}, s) = 1, \tag{17}$$

$$r(D, \mathcal{N}) - r(\mathcal{N}, D) = 0 \quad \forall D \in \mathcal{N} \setminus \{s, t\}, \tag{18}$$

$$r(\mathcal{N}, t) - r(t, \mathcal{N}) = 1. \tag{19}$$

$r(D, \mathcal{N})$ is the sum of *outflows* from D; $r(\mathcal{N}, D)$ is the sum of *inflows* to D. Thus (17) means the net outflow from s is one; (18) means the inflows equal to the outflows at each node $D \notin \{s, t\}$; (19) means the net inflows to t is one.

The following lemma provides a necessary and sufficient condition for the existence of a nonnegative flow satisfying some capacity constraints. For each arc (D, E), let $l(D, E)$ and $u(D, E)$ be exogenously given lower and upper bounds of the flow $r(D, E)$ of the arc. We prove the result using the maximum-flow theorem from Ford and Fulkerson (2015). We provide the proof in section F.2 of the online appendix.[21]

Lemma B.5. *Let $l, u : \mathcal{A} \to \mathbf{R}_+$ be such that $l(D, E) \leq u(D, E)$ for $(D, E) \in \mathcal{A}$. There exists a flow $r : \mathcal{A} \to \mathbf{R}_+$ such that*

$$l(D, E) \leq r(D, E) \leq u(D, E) \quad \forall (D, E) \in \mathcal{A} \tag{20}$$

if and only if the following condition holds for all $\mathcal{C} \subseteq \mathcal{N}$

$$\sum_{(D,E) \in \mathcal{C} \times \mathcal{C}^c} u(D, E) - \sum_{(D,E) \in \mathcal{C}^c \times \mathcal{C}} l(D, E) \geq \begin{cases} 1 & \textit{if } t \notin \mathcal{C}, s \in \mathcal{C}, \\ -1 & \textit{if } t \in \mathcal{C}, s \notin \mathcal{C}, \\ 0 & \textit{otherwise.} \end{cases} \tag{21}$$

To interpret condition (21), remember that for any collection $\mathcal{C} \subseteq \mathcal{N}$, $r(D, E)$ is called an *outflow* from $\mathcal{C}$ if $D \in C$ and $E \notin \mathcal{C}$; $r(D, E)$ is called an *inflow* to $\mathcal{C}$

[20] We define $r(D, \mathcal{N}) = 0$ if $(D, E) \notin \mathcal{A}$ for any $E \in \mathcal{N}$; Similarly, $r(\mathcal{N}, D) = 0$ if $(E, D) \notin \mathcal{A}$ for any $E \in \mathcal{N}$.

[21] We appreciate prof. Ui who pointed out a similar result appears in Rockafellar (1998).

if $D \notin C$ and $E \in \mathcal{C}$. Thus the left-hand side is the sum of the upper bounds of outflows from $\mathcal{C}$ minus the sum of the lower bounds of inflows to $\mathcal{C}$.[22] On the other hand, the right-hand side is the net outflow from $\mathcal{C}$.

We now prove Lemma B.3 by using Lemma B.5. To apply Lemma B.5 to our setup, let

$$
\begin{aligned}
&\mathcal{N} = 2^X, \quad \mathcal{A} = \{(D, D \cup x) \mid D \subseteq X, x \notin D\}, \ s = \emptyset, \quad t = X, \\
&l(D, D \cup x) = u(D, D \cup x) = K(\rho, D \cup x, x) \text{ if } (D \cup x, x) \in \mathcal{M}, \\
&l(D, D \cup x) = 0 \text{ and } u(D, D \cup x) = +\infty \ \text{ if } (D \cup x, x) \notin \mathcal{M}.
\end{aligned} \tag{22}
$$

Under the setup (22), there exists a solution r to (P2) $\Leftrightarrow$ there exists a flow r that satisfies the conditions in Lemma B.5 $\Leftrightarrow$ the condition (21) holds for any $\mathcal{C} \subseteq \mathcal{N}$, where the first equivalence holds by the setup and the second equivalence holds by Lemma B.5. Thus, to show the lemma, it suffices to prove that the condition (21) holds for any $\mathcal{C} \subseteq \mathcal{N}$ if and only if $\delta_\rho(\mathcal{C}) \geq 0$ for any complete collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$. We show the details in the online appendix. Here, we observe the following claim, which is a part of one direction of the proof.

Claim: Suppose that (21) holds for any $\hat{\mathcal{C}} \subseteq \mathcal{N}$. For any $\mathcal{C} \subseteq \mathcal{N}$ such that all outflow from $\mathcal{C}$ is observable, then $\delta_\rho(\mathcal{C}) \geq 0$.

Proof. Fix any $\mathcal{C} \subseteq \mathcal{N}$. Note that

$$\sum_{(D,E)\in\mathcal{C}^c\times\mathcal{C}} l(D, E) = \sum_{(E,y):E\notin\mathcal{C}, E\cup y\in\mathcal{C}, (E\cup y, y)\in\mathcal{M}} K(\rho, E \cup y, y).$$

Assume that any outflow from $\mathcal{C}$ is observable. Then u does not take the value of $+\infty$. Thus we have

$$\sum_{(D,E)\in\mathcal{C}\times\mathcal{C}^c} u(D, E) = \sum_{(D,x):D\in\mathcal{C}, D\cup x\notin\mathcal{C}, (D\cup x, x)\in\mathcal{M}} K(\rho, D \cup x, x).$$

Thus the left-hand side minus the right-hand side of (21) equals to the value of $\delta_\rho(\mathcal{C})$. By the suppostion of the statement that (21) holds for any $\hat{C} \subset \mathcal{N}$, we have $\delta_\rho(\mathcal{C}) \geq 0$. ■

[22] In network-flow theory, this value is called residual capacity of a cut $(\mathcal{C}, \mathcal{C}^c)$.

B.3 Proof of Lemma B.4

We prove Lemma B.4 by proving the following two lemmas. The first lemma shows that checking all test collections belong to $\mathcal{D}$, rather than all complete collections, is enough.

Lemma B.6. *If $\delta_\rho(\mathcal{C}) \geq 0$ is for any test collection $\mathcal{C} \subseteq \mathcal{D}$, then $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$.*

This lemma reduces the number of conditions to be checked because any test collection is complete and it allows us to focus only on $\mathcal{D}$. The next lemma shows that we do not have to check $\delta_\rho(\mathcal{C}) \geq 0$ for the nonessential test collections.

Lemma B.7. *Let $\mathcal{C} \equiv \{A \cup E \mid E \in \mathcal{E}\}$ be a test collection with $A \subseteq \tilde{X}$ and $\mathcal{E} \subseteq 2^{X^*}$. Assume that $\mathcal{C} \subseteq \mathcal{D}$. (i) If $\mathcal{E} = 2^{X^*}$, then $\delta_\rho(\mathcal{C}) = 0$; (ii) Suppose $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$. If $A = \tilde{X}$ or $A = \emptyset$, then $\delta_\rho(\mathcal{C}) \geq 0$.*

Combining Lemma B.3, Lemma B.6, and Lemma and B.7 immediately imply that checking the essential test collections, rather than all test collections, is enough, which is the statement of Lemma B.4.

B.3.1 Proof of Lemma B.6

We will prove the statement by the following two steps.

Step 1: If $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any test collection $\hat{\mathcal{C}} \subseteq \mathcal{D}$, then $\delta_\rho(\mathcal{C}) \geq 0$ for any test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$.

Proof. Fix a test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$. Assume that $\mathcal{C} \nsubseteq \mathcal{D}$.

Case 1: Suppose that $\mathcal{C} \cap \mathcal{D} = \emptyset$. By the property of $\mathcal{D}$ (i.e., $D \in \mathcal{D}$ & $E \supseteq D \implies E \in \mathcal{D}$), there are no observable inflow into $\mathcal{C}$. That is, if there exists (E, y) such that $E \notin \mathcal{C}$ and $E \cup y \in \mathcal{C}$, then $E \cup y \notin \mathcal{D}$, which shows that $\delta_\rho(\mathcal{C})$ does not contain any $-\sum_{(E,y):E\notin\mathcal{C},E\cup y\in\mathcal{C},E\cup y\in\mathcal{D},y\in\tilde{X}} K(\rho, E \cup y, y)$. Moreover, since $\emptyset \notin \mathcal{C}$, $\delta_\rho(\mathcal{C})$ does not contain $-1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}$ either. This means that $\delta_\rho(\mathcal{C})$ does not contain any negative terms. Thus $\delta_\rho(\mathcal{C}) \geq 0$.

Case 2: Suppose that $\mathcal{C} \cap \mathcal{D} \neq \emptyset$. Let $\mathcal{C}^* = \mathcal{C} \cap \mathcal{D}$. By the property of $\mathcal{D}$ (i.e., $D \in \mathcal{D}$ & $E \supseteq D \implies E \in \mathcal{D}$), $\mathcal{D}$ is complete. Since $\mathcal{C}$ is complete, its union $\mathcal{C}^*$

is also complete. Since $\mathcal{C}^* \subseteq \mathcal{D}$, it follows from our supposition that $\delta_\rho(\mathcal{C}^*) \geq 0$. By the disjoint additivity of δ_ρ (Lemma B.2), $\delta_\rho(\mathcal{C}) = \delta_\rho(\mathcal{C}^*) + \sum_{D \in \mathcal{C} \setminus \mathcal{C}^*} \delta_\rho(\{D\})$. Since for any $D \in \mathcal{C} \setminus \mathcal{C}^* \equiv \mathcal{C} \cap \mathcal{D}^c$, we have $\{D\} \cap \mathcal{D} = \emptyset$. Since $D \neq \emptyset$, by Case 1, we have $\delta_\rho(\{D\}) \geq 0$. Thus, we have $\delta_\rho(\mathcal{C}) \geq 0$. ■

Step 2: If $\delta_\rho(\mathcal{C}) \geq 0$ is for any test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$, then $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$.

Proof. Fix a complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$. Decompose $\hat{\mathcal{C}}$ as follows: for each $A \subseteq \tilde{X}$ write $\mathcal{C}_A = \{D \in \hat{\mathcal{C}} : D \setminus X^* = A\}$. Clearly $\hat{\mathcal{C}} = \bigcup_{A \subseteq \tilde{X}} \mathcal{C}_A$. It is easy to see that each $\mathcal{C}_A$ is a test collection and $\emptyset \notin \mathcal{C}_A$. Thus by the assumption of the step, $\delta_\rho(\mathcal{C}_A) \geq 0$. Notice that for $A \neq B$, $\mathcal{C}_A$ and $\mathcal{C}_B$ are disjoint. By Lemma B.2, $\delta_\rho(\mathcal{C})$ can be written as $\delta_\rho(\hat{\mathcal{C}}) = \sum_{A \subseteq \tilde{X}} \delta_\rho(\mathcal{C}_A) \geq 0$. ■

B.3.2 Proof of Lemma B.7

Let $\mathcal{C} \equiv \{A \cup E \mid E \in \mathcal{E}\}$ be a test collection with $A \subseteq \tilde{X}$ and $\mathcal{E} \subseteq 2^{X^*}$. Assume that $\mathcal{C} \subseteq \mathcal{D}$.

Step 1: If $\mathcal{E} = 2^{X^*}$, then $\delta_\rho(\mathcal{C}) = 0$.

Proof. By the fact that $\mathcal{E} = 2^{X^*}$ and $\mathcal{C} \subseteq \mathcal{D}$, all flows into and out of $\mathcal{C}$ are observable.[23] By the equality of inflows and outflows (not necessarily nonnegative), it follows that $\delta_\rho(\mathcal{C})$ is zero. ■

Step 2: Suppose $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$. If $A = \tilde{X}$ or $A = \emptyset$, then $\delta_\rho(\mathcal{C}) \geq 0$.

Proof. Assume $A = \tilde{X}$. Fix any $D \in \mathcal{C}$ such that $D \neq X$. By the supposition, there is no observable flows coming out from D. Since $K(\rho, D, x) \geq 0$ for all $x \in \tilde{X}$, it follows from the definition of δ_ρ, $\delta_\rho(D) \leq 0$ for all $D \neq X$. Remember $\delta_\rho(\mathcal{C}) = \sum_{D \in \mathcal{C}} \delta_\rho(D)$. Since $\delta_\rho(D) \leq 0$ for all $D \in \mathcal{C} \setminus x$, it suffices to prove $\delta_\rho(\mathcal{C}) = 0$ where $\mathcal{C}$ is the largest, or $\mathcal{C} = \{\tilde{X} \cup E \mid E \in 2^{X^*}\}$. By Step 1, we have $\delta_\rho(\{\tilde{X} \cup E \mid E \in 2^{X^*}\}) = 0$.

Assume $A = \emptyset$. If $\emptyset \in \mathcal{C}$, then $\mathcal{C} = 2^{X^*}$ by the fact that $\mathcal{C}$ is complete.

[23] That is, (i) if there exists (D, x) such that $D \in \mathcal{C}$ and $(D, x) \notin \mathcal{C}$, then $D \in \mathcal{D}$; and (ii) if there exists (E, y) such that $E \notin \mathcal{C}$ and $E \cup y \in \mathcal{C}$, then $E \cup y \in \mathcal{D}$.

Thus, all inflows into $\mathcal{C}$ are not observable and all outflows from $\mathcal{C}$ are observable, $\delta_\rho(\mathcal{C}) = \sum_{(D,x):D\setminus x\in\mathcal{C},D\notin\mathcal{C}} K(\rho, D, x) \geq 0$. ■

C Proof of Statement (b) of Theorem 3.2

First, we explain the outline of the proof of statement (b). In particular, we show the following two statements, which imply statement (b).

(1) For each $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$, there exists an incomplete dataset $\rho \in \mathbf{R}^{\mathcal{M}}$ such that (a) $K(\rho, D, x) < 0$; and $K(\rho, E, y) \geq 0$ for all $(E, y) \in \mathcal{M} \setminus \{(D, x)\}$; (b) $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$.

(2) For each essential test collection $\mathcal{C}^* \subseteq \mathcal{D}$, there exists an incomplete dataset $\rho \in \mathbf{R}^{\mathcal{M}}$ such that (a) $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$;(b) $\delta_\rho(\mathcal{C}^*) < 0$ and $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$.

To prove the two statements above, we show the corresponding statements in terms of flows assuming $\mathcal{D} = 2^X \setminus \emptyset$. (See Lemma C.2 below.) Then, we convert the flows from into a complete dataset by using the next lemma:

Lemma C.1. *Let $\mathcal{D} = 2^X \setminus \emptyset$. If there exists $r \in \mathbf{R}^{\{(D\setminus x,D)|x\in D\in 2^X\}}$ satisfying the following three conditions:*

(i) $\sum_{x\in X} r(X \setminus x, X) = 1$;

(ii) for any $D \in \mathcal{D}$ s.t. $1 \leq |D| < |X|$, $\sum_{x\in D} r(D \setminus x, D) = \sum_{y\notin D} r(D, D \cup y)$;

(iii) for any $x \in D \in \mathcal{D}$, $\sum_{E:E\supseteq D} r(E \setminus x, E) \geq 0$,

then there exists a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ such that $\sum_{x\in D} \rho^*(D, x) = 1$ for all $D \in D$ and $K(\rho^*, D, x) = r(D\setminus x, D)$ for any (D, x) such that $x \in D \in 2^X$.

We provide the proof of Lemma C.1 in Section F.3 in the online appendix.

We now introduce a new notation that corresponds to the definition (16) of δ_ρ: for any $\mathcal{C} \subseteq \mathcal{N}$, define[24]

$$\delta_r(\mathcal{C}) = \left(\sum_{\substack{(D,x):\\ D\in\mathcal{C}, D\cup x\notin\mathcal{C}, x\in\tilde{X}}} r(D, D\cup x) - \sum_{\substack{(E,y):\\ E\notin\mathcal{C}, E\cup y\in\mathcal{C}, y\in\tilde{X}}} r(E, E\cup y) \right)$$
$$+1\{X\in\mathcal{C}, \emptyset\notin\mathcal{C}\} - 1\{\emptyset\in\mathcal{C}, X\notin\mathcal{C}\}. \tag{23}$$

Lemma C.2. *Let* $\mathcal{D} = 2^X \setminus \emptyset$.

(i) For each $(D,x) \in \mathcal{M}$ *such that* $1 < |D| < |X|$, *there exists* $r \in \mathbf{R}^{\{(D\setminus x, D)|x\in D\in\mathcal{D}\}}$ *that satisfies conditions (i)–(iii) in Lemma C.1 and the following two conditions: (a)* $r(D\setminus x, D) < 0$; $r(E\setminus y, E) \geq 0$ *for all* $(E,y)\in\mathcal{M}$ *s.t. and* $(E,y) \neq (D,x)$*; (b)*$\delta_r(\mathcal{C}) \geq 0$ *for any essential test collection* $\mathcal{C}$*.*

(ii) For each essential test collection $\mathcal{C}^*$, *there exists* $r \in \mathbf{R}^{\{(D\setminus x, D)|x\in D\in\mathcal{D}\}}$ *that satisfies conditions (i)–(iii) in Lemma C.1 and the following two conditions: (a)* $r(D\setminus x, D) \geq 0$ *for all* $x\in\tilde{X}$*; (b)* $\delta_r(\mathcal{C}^*) < 0$ *and* $\delta_r(\mathcal{C}) \geq 0$ *for any essential test collection* $\mathcal{C}$ *except* $\mathcal{C}^*$*.*

We will provide the proof of statement Lemma C.2 in the next section.

Given Lemma C.1 and C.2, we now prove statement (1) and (2) as follows:

We will prove statement (2), fix $\mathcal{C}^* \subseteq \mathcal{D}$. By Lemma C.2 (ii), there exists $r \in \mathbf{R}^{\{(D\setminus x, D)|x\in D\in 2^X\}}$ that satisfies conditions (i)–(iii) in Lemma C.1 and conditions (a) and (b) in Lemma C.2 (ii). By Lemma C.1, there exists a complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ such that $\sum_{x\in D}\rho^*(D,x) = 1$ for all $D\in 2^X$ and $K(\rho^*, D, x) = r(D\setminus x, D)$ for any (D,x) such that $x\in D\in 2^X$. Let ρ be the restriction of ρ^* to $\mathcal{M}$. In the following, we will show that $\delta_\rho(\mathcal{C}^*) < 0$ and $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$.

Let δ_{ρ^*} be the function defined by (16) with respect to the complete dataset

[24] Conditions after the summations are different between δ_ρ and δ_r because we are considering the complete dataset here. When $\mathcal{D} = 2^X \setminus \emptyset$, both conditions are equivalent.

ρ^* with $\mathcal{D} = 2^X$ and $X^* = \emptyset$.[25] Let δ_ρ be the function defined by (16) with respect to the incomplete dataset of ρ with given $\mathcal{D}$ and X^*. Remember δ_r is the function defined after Lemma C.1. Note that for any test collection $\mathcal{C} \subseteq \mathcal{D}$, $\delta_r(\mathcal{C}) = \delta_{\rho^*}(\mathcal{C}) = \delta_\rho(\mathcal{C})$, where the first equality holds because $K(\rho^*, D, x) = r(D \setminus x, D)$ and the second equality holds because the value of δ does not depend on the values of ρ^* and ρ outside of $\mathcal{M}$; and $\rho = \rho^*$ on $\mathcal{M}$. Thus, we have $\delta_\rho(\mathcal{C}^*) < 0$ and $\delta_\rho(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$.

Statement (1) can be proved exactly in the same way using Lemma C.2 (i) instead of (ii).

C.1 Proof of Statement (i) in Lemma C.2

Let $\mathcal{H} \equiv \{D \in \mathcal{D} \mid \text{there exists an essential test collection } \mathcal{C} \text{ such that } D \in \mathcal{C}\}$. Explicitly, $\mathcal{H}$ is the set of all $D \in \mathcal{D}$ which have nonempty intersection with X^*, nonempty intersection with $\tilde{X}$, and do not contain $\tilde{X}$. Note that for the purpose of this construction, we may ignore any parts of the flow which do not pass through $\mathcal{H}$ since these parts do not change $\delta_r(\mathcal{C})$ for any essential test collection $\mathcal{C}$. We try to consider flows which pass through $\mathcal{H}$ as little as possible to make the construction as simple as possible.

Claim C.3. *There exists a $\emptyset - X$ directed path Π_1 avoiding all nodes in $\mathcal{H}$.*

We provide the proof of the claim in the online appendix. For any $\emptyset - X$ directed path, denote by r^Π the flow which puts weight one on all arcs in Π and zero otherwise.

To prove statement (i), choose any arc $(D \setminus x, D)$ with $x \in \tilde{X}$, $D \setminus x \neq \emptyset$, $D \neq X$. We will consider two cases:

Case 1: $D^c \cap \tilde{X} \neq \emptyset$. Let Π_1 be a $\emptyset$ to X dipath which avoids any nodes in $\mathcal{H}$. (Such a dipath exists by Claim C.3.) Let Π_2 be a dipath from $\emptyset$ to D which passes through $D \cap \tilde{X}$. Let Π_3 be a $D \setminus x$ to X dipath which passes through $D \cup \tilde{X}$ but not D. Such a dipath exists because $D^c \cap \tilde{X} \neq \emptyset$ implies that there exists an observable

[25] That is, $\delta_{\rho^*}(\mathcal{C}) = \left(\sum_{(D,x):D\in\mathcal{C}, D\cup x\notin\mathcal{C}} K(\rho^*, D \cup x, x) - \sum_{(E,y):E\notin\mathcal{C}, E\cup y\in\mathcal{C}} K(\rho^*, E \cup y, y)\right) + 1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}$.

alternative $y \in D^c \cap \tilde{X}$ and there exists an arc $(D \setminus x, D \cup y \setminus x)$. Fix $\varepsilon > 0$ and define $r^* \equiv (1-\varepsilon)r^{\Pi_1} + \varepsilon r^{\Pi_2} - \varepsilon r^{(D\setminus x,D)} + \varepsilon r^{\Pi_3}$, where $r^{(D\setminus x,D)}$ is a vector that gives one only at the arc $(D \setminus x, D)$; zero elsewhere. By definition, $r^*(D \setminus x, D) < 0$ and $r^*(E \setminus y, E) \geq 0$ for any (E, y) such that $y \in \tilde{X}$ and $(D, x) \neq (E, y)$. Moreover, for any essential test collection $\mathcal{C}$, $\delta_{r^*}(\mathcal{C}) \geq 0$. To see this notice that the flow r^{Π_1} does not change any value of $\delta_r(\mathcal{C})$ for any essential test collection. By the definition of r^*, we have $\delta_{r^*}(\{D \setminus x\}) = 0$ and $\delta_{r^*}(\{D\}) \geq 0$.[26] For all other nodes E, $\delta_{r^*}(\{E\}) = 0$. Thus, we have $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.

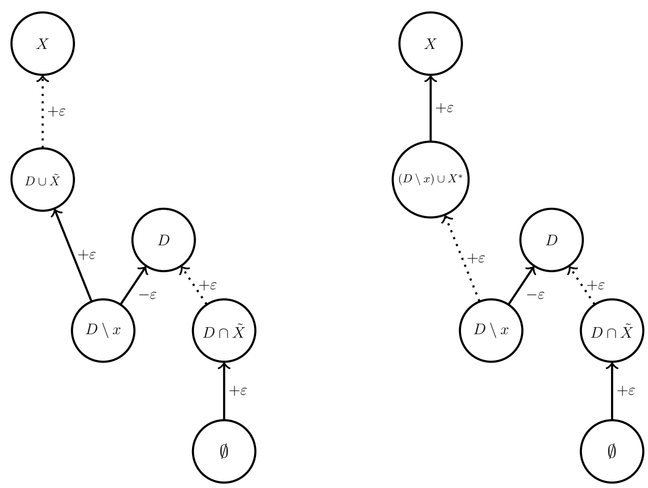

Figure 5: Flow $\varepsilon r^{\Pi_2} - \varepsilon r^{(D\setminus x,D)} + \varepsilon r^{\Pi_3}$ in Case 1 (left); Flow $\varepsilon r^{\Pi_2} - \varepsilon r^{(D\setminus x,D)} + \varepsilon r^{\Pi_4}$ in Case 2 (right) of section C.1

Note: Given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M}}$, solid arrows correspond to observable flows and dotted arrows correspond to unobservable flows. Note also that all nodes other than D appearing in the figure do not belong to $\mathcal{H}$; thus the values of δ_r are zero on $\mathcal{H} \setminus D$.

Case 2: $D^c \cap \tilde{X} = \emptyset$. This means that D contains all elements in X^*. Notice $(D \setminus x) \cap X^* \neq X^*$ since otherwise $D = X$. So let Π_4 be a $D \setminus x$ to X dipath that passes through $(D \setminus x) \cup X^*$. (Note that the last arc is observable arc $(X \setminus x, X)$, where $x \in \tilde{X}$.) Define $r^* \equiv (1-\varepsilon)r^{\Pi_1} + \varepsilon r^{\Pi_2} - \varepsilon r^{(D\setminus x,D)} + \varepsilon r^{\Pi_4}$. By definition, $r^*(D \setminus x, D) < 0$ and $r^*(E \setminus y, E) \geq 0$ for any (E, y) such that $y \in \tilde{X}$ and $(D, x) \neq (E, y)$. Moreover, for any essential test collection $\mathcal{C}$, $\delta_{r^*}(\mathcal{C}) \geq 0$. To see this notice (i) $\delta_{r^*}(\{D \setminus x\}) = -\varepsilon$ but $\delta_{r^*}(\{(D \setminus x) \cup X^*\}) = \varepsilon$; (ii) any test

[26] $\delta_{r^*}(\{D\})$ is either 0 or ε.

collection $\mathcal{C}$ containing $D \setminus x$ contains $(D \setminus x) \cup X^*$. (i) and (ii) implies that the negative value of $\delta_{r^*}(\{D \setminus x\})$ is cancelled by the positive value of $\delta_{r^*}((D \setminus x) \cup X^*)$. For all other nodes E, $\delta_{r^*}(\{E\}) = 0$. Thus, we have $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.

C.2 Proof of Statement (ii) of Lemma C.2

Fix an essential test collection $\mathcal{C}^*$. In the following, we will construct a flow r from $\emptyset$ to X such that $\delta_r(\mathcal{C}^*) < 0$ and $\delta_r(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$. Let A^* be such that $D \setminus X^* = A^*$ for all $D \in \mathcal{C}^*$. (Such A^* exists because $\mathcal{C}^*$ is a test collection.) Since $\mathcal{C}^*$ is essential, $A^* \neq \emptyset$ and $A^* \neq \tilde{X}$. Remember that $\mathcal{H} \equiv \{D \in \mathcal{D} \mid \text{there exists an essential test collection } \mathcal{C} \text{ such that } D \in \mathcal{C}\}$.

In the following we prove the following claims to prove this lemma. All proofs of the claims are in the online appendix.

Claim C.4. *Fix $\varepsilon \in (0, 1]$. For any D in $\mathcal{C}^*$, there exists a flow r_D^1 such that $\delta_{r_D^1}(\{D\}) = -\varepsilon$ and $\delta_{r_D^1}(\{H\}) = 0$ for any $H \in \mathcal{H} \setminus D$. Moreover r_D^1 satisfies the three conditions in the Lemma C.1.*

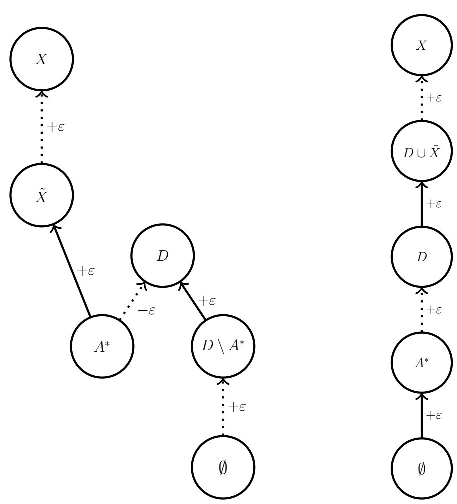

Figure 6: Flow r_D^1 in Claim C.4 (left); Flow r_D^2 in Claim C.5 (right)

Claim C.5. *Fix $\varepsilon \in (0,1]$. For any D in $\mathcal{C}^*$, there exists a flow r^2_D such that $\delta_{r^2_D}(\{D\}) = \varepsilon$ and $\delta_{r^2_D}(\{H\}) = 0$ for all $H \in \mathcal{H} \setminus D$. Moreover the flow r^2_D satisfies the all conditions in Lemma C.1.*

By considering a convex combination of the flows defined in the previous claims, we can construct r^* such that $\delta_{r^*}(\mathcal{C}^*) < 0$ and $\delta_{r^*}(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$; moreover, r^* satisfies conditions (i)–(iii) in Lemma C.1 and $r(D \setminus x, D) \geq 0$ for all $x \in \tilde{X}$ and $D \supseteq x$.[27]

D Proof of Propositions

D.1 Proof of Proposition 3.5

Suppose ρ satisfies the conditions of Proposition 3.5, that is ρ satisfies (i) and (ii) for singleton test collections. By Falmagne (1978), it suffices to show that $K(\hat{\rho}, \hat{D}, x) \geq 0$ for all $(\hat{D}, x)$ such that $x \in \hat{D} \in \hat{\mathcal{D}}$.

Case 1: $x \in \tilde{X}$.

Remember that we write $D = (\hat{D} \cap \tilde{X}) \cup X^*$ if $x_0 \in \hat{D}$ and $D = \hat{D}$ otherwise. For all $x \in \hat{D} \in \hat{\mathcal{D}}$ let $r(\hat{D}, x) = K(\rho, D, x)$ if $x_0 \in \hat{D}$ and $r(\hat{D}, x) = \sum_{E \subsetneq X^*} K(\rho, D \cup E, x)$ if $x_0 \notin \hat{D}$. For all $\hat{D} \in \hat{D}$, by the definition we have $\hat{\rho}(\hat{D}, x) = \rho(D, x)$, thus $\hat{\rho}(\hat{D}, x) = \sum_{E \supseteq D} K(\rho, E, x)$. By calculation, it is easy to see that $\sum_{E \supseteq D} K(\rho, E, x) = \sum_{\hat{E} \supseteq \hat{D}} r(\hat{E}, x)$. Thus $\hat{\rho}(\hat{D}, x) = \sum_{\hat{E} \supseteq \hat{D}} r(\hat{E}, x)$ for all $\hat{D} \in \hat{\mathcal{D}}$. Since the Block-Marschak polynomial is the unique vector with this property, it follows that $r(\hat{D}, x) = K(\hat{\rho}, \hat{D}, x)$ for all $x \in \hat{D} \in \hat{\mathcal{D}}$. $r(D, x)$ is nonnegative so we are done.

Case 2: $x = x_0$.

Fix $\hat{D}$ with $x_0 \in \hat{D}$. By the inflow outflow equality, $\sum_{y \notin \hat{D}} K(\hat{\rho}, \hat{D} \cup \{y\}, y) =$

[27] r^* can be explicitly defined as follows with the appropriate choice of α,

$$r^* = \frac{\alpha}{2}\Big(\sum_{D \in \mathcal{C}^*} \frac{1}{|\mathcal{C}^*|} r^1_D\Big) + \frac{\alpha}{2}\Big(\frac{1}{|\mathcal{C}^*|} r^{\Pi_1} + \frac{|\mathcal{C}^*| - 1}{|\mathcal{C}^*|} r^2_{A^* \cup X^*}\Big) + (1 - \alpha) \sum_{D_\mathcal{C} \in \mathcal{F}} \frac{1}{|\mathcal{F}|} r^2_{D_\mathcal{C}},$$

where Π_1 is the directed path in Claim C.3. See the details in the online appendix for the definitions of $\mathcal{F}$ and $D_\mathcal{C}$.

$\sum_{z\in\hat{D}} K(\hat{\rho}, D, z)$. Thus,

$$\begin{aligned}
K(\hat{\rho}, \hat{D}, x_0) &= \sum_{y\notin\hat{D}} K(\hat{\rho}, \hat{D}\cup\{y\}, y) - \sum_{z\in\hat{D}\cap\tilde{X}} K(\hat{\rho}, D, z) \\
&= \sum_{y\notin D} K(\rho, D\cup\{y\}, y) - \sum_{z\in D\cap\tilde{X}} K(\rho, D, z) \geq 0,
\end{aligned}$$

where the first equality is rearranging the inflow equals outflow equality, the second applies the identity from case 1, and nonnegativity follows from condition (ii) in the case $\mathcal{C} = \{D\}$.

D.2 Proof of Proposition 4.4

Let $\rho \in \mathbf{R}_+^{\mathcal{M}}$ be a given incomplete dataset. We first introduce a definition: A complete dataset $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ is said to be *RU-consistent* with ρ if it satisfies the following two conditions: (α) $\rho = \rho^*$ on $\mathcal{M}$; and (β) there exists $\mu \in \Delta(\mathcal{L})$ such that for any (D, x) with $x \in D \in 2^X$, $\rho^*(D, x) = \mu(\succ\in\mathcal{L} \mid x \succ y$ for all $y \in D\setminus x)$.

Let Γ be the set of complete datasets $\rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ that are RU-consistent with the given incomplete dataset ρ. Remember that $\rho(D, x)$ is defined (i.e., observable) if and only if $(D, x) \in \mathcal{M}$. With $(D, x) \notin \mathcal{M}$ fixed, the goal of the proof is to obtain bounds of $\rho^*(D, x)$ for some $\rho^* \in \Gamma$. As is pointed out in page 1399 of Manski (2007), since Γ is convex and conditions (α) and (β) are linear, the identified set is an interval with the upper and lower bounds given by $\overline{\rho}(D, x) \equiv \max_{\rho^*\in\Gamma} \rho^*(D, x)$ and $\underline{\rho}(D, x) \equiv \min_{\rho^*\in\Gamma} \rho^*(D, x)$, respectively.

Based on the idea of (P1) in Section 3.2, Γ can be written as

$$\left\{ \rho^* \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}} \;\middle|\; \begin{array}{l} \text{There exists } \mu \in \Delta(\mathcal{L}) \text{ satisfying} \\ \text{the conditions } (\alpha) \text{ and } (\beta) \text{ above} \end{array} \right\} \tag{24}$$

The formulation (24) is closely related to (P1): it can be shown that if μ satisfies the conditions (α) and (β), it is a solution to (P1). To obtain the proposition, we rewrite Γ in the spirit of (P2) exploiting the network flow structure.

By Möbius inversion, condition (β) above can be written as follows: for all

(D, x) such that $x \in D \in 2^X$, $\mu(\{\succ \in \mathcal{L} \mid D^c \succ x \succ D \setminus x) = K(\rho^*, D, x)$. Thus by condition (7), if r is a solution to (P2) then $\mu(\{\succ \in \mathcal{L} \mid D^c \succ x \succ D \setminus x\}) = r(D \setminus x, D)$. Therefore condition (β) is equivalent to

$$r(D \setminus x, D) = K(\rho^*, D, x) \text{ for all } (D, x) \text{ such that } x \in D \in 2^X. \tag{25}$$

Thus we can rewrite the set Γ (i.e., (24)) as follows:

$$\left\{ \rho^* \in \mathbf{R}_+^{\{(D,x) \mid x \in D \in 2^X\}} \;\middle|\; \begin{array}{l} \text{There exists a solution } r \in \mathbf{R}_+^{\{(D \setminus x, D) \mid x \in D \in 2^X\}} \text{ to} \\ \text{(P2) satisfying (25) and } \rho^* = \rho \text{ on } \mathcal{M}. \end{array} \right\} \tag{26}$$

By eliminating observable flows r (i.e., $r(D \setminus x, D) = K(\rho, D, x)$ for all $(D, x) \in \mathcal{M}$) in (P2), it can be verified that the conditions (5) and (6) of (P2) are equivalent to (15).[28] Using the Möbius inversion formula, we also can rewrite (25) into the following: for all (D, x) such that $x \in D \in 2^X$, we have

$$\rho^*(D, x) = \sum_{E : E \supseteq D} r(E \setminus x, E), \tag{27}$$

where $r(E \setminus x, E) = K(\rho, E, x)$ for all $(E, x) \in \mathcal{M}$. These observations imply that we can rewrite the set (26) into the following set:

$$\left\{ \rho^* \in \mathbf{R}_+^{\{(D,x) \mid x \in D \in 2^X\}} \;\middle|\; \begin{array}{l} \text{There exists } r \in \mathbf{R}_+^{\{(D \setminus x, D) \mid x \in D \in 2^X\}} \\ \text{satisfying (15) and (27) and } \rho^* = \rho \text{ on } \mathcal{M} \end{array} \right\}. \tag{28}$$

It follows that the upper bound becomes $\overline{\rho}(D, x) = \max_{r \in \mathbf{R}_+^{\{(D,x) \mid x \in D \in 2^X\}}} \sum_{E : E \supseteq D} r(E \setminus x, E)$ subject to (15), where $r(E \setminus x, E) = K(\rho, E, x)$ for all $(E, x) \in \mathcal{M}$. The corresponding result for the lower bound can be obtained by changing max to min.

E Summary of Liao, Saito, and Sandroni (2025)

In this section we briefly summarize the main findings of the companion paper using the notation from this paper. In the present paper, we introduce an outside

[28] Condition (7) in (P2) is implied by (25) and $\rho^* = \rho$ on $\mathcal{M}$.

option x_0 only for choice sets that contain all unobservable alternatives X^*. For any nonempty $A \subseteq \tilde{X} := X \setminus X^*$, we define the reduced dataset $\hat{\rho}$ on menus $A \cup \{x_0\}$ by

$$\hat{\rho}\big(A \cup \{x_0\}, x_0\big) \ := \ \rho\big(A \cup X^*,\, X^*\big), \tag{29}$$

where $\rho(A \cup X^*, X^*) \equiv \sum_{x \in X^*} \rho(A \cup X^*, x)$. Note that (29) is equivalent to the definition of $\hat{\rho}$ in Definition 3.4. For example, if $X^* = \{c, d\}$, then $\hat{\rho}(A \cup x_0, x_0) = \rho(A \cup \{c, d\}, \{c, d\})$.

In Liao, Saito, and Sandroni (2025), we consider a more general setting in which the composite of the outside option is described as an unknown distribution to the analyst. More precisely, for $A \subseteq \tilde{X}$ and $E \subseteq X^*$, we denote by $\lambda_{A \cup x_0}(E)$ the probability that the actual menu the decision maker faces is $A \cup E$. (In other words, the outside option x_0 represents the set E of unobservable alternatives.) We call λ the *composition distribution*. Notice that in the setup the analyst does not observe which options are included in the outside option x_0. Under this formulation, the choice frequency of the outside option is determined by two components: how likely each unobservable alternative is in the outside option and how likely one of the alternatives in the outside option is chosen from the menu. We define that $\hat{\rho}$ is *RU-rationalizable* if there exists a composition distribution λ and a distribution μ over linear orders on the observable alternatives as well as the unobservable alternatives such that for all $a \in A$:

$$\hat{\rho}(A \cup x_0, a) = \sum_{E \in 2^{X^*} \setminus \emptyset} \lambda_{A \cup x_0}(E) \mu(\succ \mid a \succ b \text{ for all } b \in (A \cup E) \setminus \{a\}). \tag{30}$$

Note that the formula (30) implies that the probability that the outside option is chosen from $A \cup x_0$ equals

$$\hat{\rho}(A \cup x_0, x_0) = \sum_{E \in 2^{X^*} \setminus \emptyset} \lambda_{A \cup x_0}(E) \rho(A \cup E, E), \tag{31}$$

where $\rho(A \cup E, E) = \sum_{x^* \in E} \rho(A \cup E, x^*)$. For instance, if $X^* = \{c, d\}$, then $\hat{\rho}(A \cup x_0, x_0) = \lambda_{A \cup x_0}(\{c, d\}) \rho(A \cup \{c, d\}, \{c, d\}) + \lambda_{A \cup x_0}(\{c\}) \rho(A \cup \{c\}, c) + \lambda_{A \cup x_0}(\{d\}) \rho(A \cup \{d\}, d)$. Note also that when $\lambda_{A \cup x_0}(X^*) = 1$, this formula (31) coincides with (29).

One of the main results in the companion paper shows that $\hat{\rho}$ is RU-rationalizable if and only if: (i) $\hat{\rho}$ is RU-rationalizable on all choice sets that do not include the outside option; (ii) For every choice set E that does not contain x_0, and for all $y \in E$, we have $\hat{\rho}(E, y) \geq \hat{\rho}(E \cup x_0, y)$.

This result shows that the outside option approach imposes extremely weak constraints on the data. The importance of this result as it relates to the present paper is that these two implications are *not* enough to guarantee the existence of a distribution $\hat{\mu}$ over linear orders on $\hat{X}$ that rationalizes the reduced dataset.

The companion paper also shows that the naive approach studied in the present paper—in which λ is constant across choice sets—serves as an idealized benchmark where the model retains relatively strong implications in the sense that it guarantees the existence of a distribution $\hat{\mu}$ that rationalizes the reduced dataset. As such, our current setup provides a natural starting point for analyzing the limitations of the outside option approach. In the present paper, we show that even under this idealized scenario, key implications of the RUM may still be lost.

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Online Appendix

F Omitted Proofs

F.1 Details of Proof of Lemma B.3

Step 1: If the condition (21) holds for any $\hat{\mathcal{C}} \subseteq \mathcal{N}$, then $\delta_\rho(\mathcal{C}) \geq 0$ for any complete collection $\mathcal{C} \subseteq \mathcal{N}$ such that $\emptyset \notin \mathcal{C}$.

Proof. Fix any complete collection $\mathcal{C} \subseteq \mathcal{N}$ such that $\emptyset \notin \mathcal{C}$. By the disjoint additivity, we have $\delta_\rho(\mathcal{C}) = \delta_\rho(\mathcal{C} \cap \mathcal{D}) + \delta_\rho(\mathcal{C} \setminus \mathcal{D})$.

Since $\mathcal{D}$ is an upper set and $\mathcal{C}$ is complete, there are no unobservable outflows from $\mathcal{C} \cap \mathcal{D}$. Thus by the supposition of this step, it follows from Claim in the body of the paper that $\delta_\rho(\mathcal{C} \cap \mathcal{D}) \geq 0$.

Since $\emptyset \notin \mathcal{C} \setminus \mathcal{D}$ and $\mathcal{C} \setminus \mathcal{D}$ has no observable inflows, by the definition of δ_ρ, there are no negative terms in $\delta_\rho(\mathcal{C} \setminus \mathcal{D})$. Thus, we have $\delta_\rho(\mathcal{C} \setminus \mathcal{D}) \geq 0$. It follows that $\delta_\rho(\mathcal{C}) \geq 0$. ■

Step 2: If $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}} \subseteq \mathcal{N}$ such that $\emptyset \notin \hat{\mathcal{C}}$, then the condition (21) holds for any $\mathcal{C} \subseteq \mathcal{N}$.

Proof. Fix any $\mathcal{C} \subseteq \mathcal{N}$. If $\mathcal{C}$ has an unobservable outflow, then the left-hand side of (21) becomes infinite and (21) holds as desired. In the following consider the case where $\mathcal{C}$ has no unobservable outflows (i.e., all outflows are observable), which implies $\mathcal{C}$ is complete. As in the proof of Step 1, this implies that the left-hand side minus the right-hand side of (21) equals to $\delta_\rho(\mathcal{C})$.

If $\emptyset \notin \mathcal{C}$, by the supposition of this step, $\delta_\rho(\mathcal{C}) \geq 0$ because $\mathcal{C}$ is complete. Thus, (21) holds.

Assume $\emptyset \in \mathcal{C}$ in the following. Since $\mathcal{C}$ is complete, $2^{X^*} \subseteq \mathcal{C}$.

Now let $\mathcal{G}$ be the intersection of complete collections of menus containing $\mathcal{C} \cap \mathcal{D}^c$. Note that $\mathcal{G}$ is complete (the intersection of complete collections is complete) and a subset of $\mathcal{C}$ ($\mathcal{C}$ is a complete collection containing $\mathcal{C} \cap \mathcal{D}^c$). Also note that

$\emptyset \in \mathcal{G}$ because $\emptyset \in \mathcal{C}$ by our previous assumption and $\emptyset \in \mathcal{D}^c$ by definition.

Note that $\mathcal{C} \cap \mathcal{D}^c$ is also a lower set. To see why, first note that $\mathcal{C} \cap \mathcal{D}^c$ is a lower set. To see why, suppose it is not a lower set. Then there exists $E \in \mathcal{C} \cap \mathcal{D}^c$ and $E' \subset E$ such that $E' \notin \mathcal{C} \cap \mathcal{D}^c$. Since $\emptyset \in \mathcal{C} \cap \mathcal{D}^c$, somewhere on the path from $\emptyset$ to E' there is an arc coming from a node $D \in \mathcal{C} \cap \mathcal{D}^c$ to a node $D \cup x \notin \mathcal{C} \cap \mathcal{D}^c$. Since $E \in \mathcal{D}^c$, it follows that also E', D, and $D \cup x$ are all elements of $\mathcal{D}^c$. Thus the arc $(D, D \cup x)$ is an outflow from $\mathcal{C} \cap \mathcal{D}^c$. Since $D \cup x$ is in $\mathcal{D}^c$, it cannot be in $\mathcal{C}$, thus $(D, D \cup x)$ is an unobservable outflow from $\mathcal{C}$. This contradicts our assumption on $\mathcal{C}$.

Since $\mathcal{C} \cap \mathcal{D}^c$ is a lower set, so is $\mathcal{G}$. To see why, note $\mathcal{G} = \{D \cup E | D \in \mathcal{C} \cap \mathcal{D}^c, E \in 2^{X^*}\}$. If $D \subseteq E \in \mathcal{G}$, then any subset of $D \subseteq E$ can be written as $D' \cup E'$ where $D' \subseteq D$ and $E' \subseteq E$. Because $\mathcal{C} \cap \mathcal{D}^c$ is a lower set, $D' \in \mathcal{C} \cap \mathcal{D}^c$. Similarly because 2^{X^*} is a lower set $E' \in 2^{X^*}$. It follows that $D' \cup E' \in \mathcal{G}$.

Furthermore $\mathcal{G}$ has no unobservable outflow. To see why, suppose otherwise, that is $(D, D \cup x)$ is unobservable, $D \in \mathcal{G}$, and $D \cup x \in \mathcal{G}^c$. Since $\mathcal{G}$ is complete, $x \in \tilde{X}$, thus $D \cup x \in \mathcal{D}^c$. Therefore $D \cup x \notin \mathcal{C}$. But then $(D, D \cup x)$ is an unobservable outflow from $\mathcal{C}$ which does not exist by assumption.

Therefore the collection $\mathcal{G}$ has no inflows and, no unobservable outflow, and contains $\emptyset$. By the inflow outflow equality, $\delta_\rho(\mathcal{G}) = 0$.

Note that $\mathcal{C} \setminus \mathcal{G} = \mathcal{C} \cap \mathcal{G}^c$ is complete because $\mathcal{G}$ is a lower set (in particular $\mathcal{G}^c$ is an upper set and thus is complete) and the intersection of complete sets is complete. Also, $\mathcal{C} \setminus \mathcal{G}$ does not contain $\emptyset$. It follows from the supposition of the step that $\delta_\rho(\mathcal{C} \setminus \mathcal{G}) \geq 0$.

Since $\delta_\rho(\mathcal{C}) = \delta_\rho(\mathcal{C} \setminus \mathcal{G}) + \delta_\rho(\mathcal{G})$, we have $\delta_\rho(\mathcal{C}) \geq 0$, which means the inequality (21) for $\mathcal{C}$. ■

F.2 Proof of Lemma B.5

To prove the lemma we prove the following general result:

Theorem F.1. *Let $T, S \subseteq \mathcal{N}$ such that $S \cap T = \emptyset$, $a : S \to \mathbf{R}_+$, $b : T \to \mathbf{R}_+$ such that $\sum_{s \in S} a(s) = 1 = \sum_{t \in T} b(t)$. There exists $r : \mathcal{A} \to \mathbf{R}_+$ such that*

$\forall s \in S[r(s,\mathcal{N}) - r(\mathcal{N},s) = a(s)]$, $\forall D \in \mathcal{N} \setminus (S \cup T)[r(D,\mathcal{N}) - r(\mathcal{N},D) = 0]$, $\forall t \in T[r(\mathcal{N},t) - r(t,\mathcal{N}) = b(t)]$, $\forall (D,E) \in \mathcal{A}[l(D,E) \leq r(D,E) \leq u(D,E)]$ *if and only if the following conditions hold for any* $\mathcal{C} \subseteq \mathcal{N}$*:*

$$\sum_{(D,E)\in\mathcal{C}\times\mathcal{C}^c} u(D,E) - \sum_{(D,E)\in\mathcal{C}^c\times\mathcal{C}} l(D,E) \geq \sum_{t\in\mathcal{C}^c\cap T} b(t) - \sum_{s\in\mathcal{C}^c\cap S} a(s). \tag{32}$$

Proof.

Necessity: Suppose a feasible flow r exists. $\sum_{t\in\mathcal{C}^c\cap T} b(t) - \sum_{s\in\mathcal{C}^c\cap S} a(s) \leq r(\mathcal{C},\mathcal{C}^c) - r(\mathcal{C}^c,\mathcal{C}) \leq \sum_{(D,E)\in\mathcal{C}\times\mathcal{C}^c} u(D,E) - \sum_{(D,E)\in\mathcal{C}^c\times\mathcal{C}} l(D,E)$.

Sufficiency: Define an extended network with lower bound by $\mathcal{N}^* = \mathcal{N} \cup \{s^*, t^*\}$ and $\mathcal{A}^* = \mathcal{A} \cup \{(s^*, s) \mid s \in S\} \cup \{(t, t^*) \mid t \in T\}$ and $u^*(s^*, s) = a(s)$, $l^*(s^*, s) = 0$ for all $s \in S$, $u^*(t, t^*) = b(t), l^*(t, t^*) = 0$ for all $t \in T$, $u^*(D, E) = u(D, E), l^*(D, E) = l(D, E)$ for all other arcs.

We first define the residual capacity function e in the augmented network: for any $\mathcal{C}^* \subseteq \mathcal{N}^*$, $e(\mathcal{C}^*, \mathcal{N}^*\backslash\mathcal{C}^*) = \sum_{(D,E)\in\mathcal{C}^*\times(\mathcal{N}^*\backslash\mathcal{C}^*)} u^*(D,E) - \sum_{(D,E)\in(\mathcal{N}^*\backslash\mathcal{C}^*)\times\mathcal{C}^*} l^*(D,E)$.[29] (Similarly, we define the residual capacity function in the original network as follows: for any $\mathcal{C} \subseteq \mathcal{N}$ $e(\mathcal{C}, \mathcal{N}\backslash\mathcal{C}) = \sum_{(D,E)\in\mathcal{C}\times(\mathcal{N}\backslash\mathcal{C})} u^*(D,E) - \sum_{(D,E)\in(\mathcal{N}\backslash\mathcal{C})\times\mathcal{C}} l^*(D,E)$.)

Then we will prove that $(\mathcal{N}^* \setminus \{t^*\}, t^*)$ is a minimum s^*-t^* cut. Let be any $\mathcal{C}^* \subseteq \mathcal{N}^*$ such that $s^* \in \mathcal{C}^*$ and $t^* \notin \mathcal{C}^*$. That is, $(\mathcal{C}^*, \mathcal{N}^* \setminus \mathcal{C}^*)$ be an arbitrary cut separating s^* and t^*. Let $\mathcal{C} = \mathcal{C}^* \cap \mathcal{N}$. Then by the structure of network,

$$\begin{aligned}
&e(\mathcal{C}^*, \mathcal{N}^* \setminus \mathcal{C}^*) - e(\mathcal{N}^* \setminus \{t^*\}, t^*) \\
&= e(s^*, \mathcal{N}^* \setminus \mathcal{C}^*) + e(\mathcal{C}, \mathcal{N}^* \setminus \mathcal{C}^*) - e(\mathcal{N}^* \setminus \{t^*\}, t^*) \quad (\because s^* \in \mathcal{C}^*) \\
&= e(s^*, \mathcal{N}^* \setminus \mathcal{C}^*) + e(\mathcal{C}, \mathcal{N} \setminus \mathcal{C}) + e(\mathcal{C}, t^*) - e(\mathcal{N}^* \setminus \{t^*\}, t^*) \quad (\because t^* \in \mathcal{N}^* \setminus \mathcal{C}^*) \\
&= e(s^*, (\mathcal{N} \setminus \mathcal{C}) \cap S) + e(\mathcal{C}, \mathcal{N} \setminus \mathcal{C}) + e(\mathcal{C} \cap T, t^*) - e(T, t^*) \\
&= e(s^*, (\mathcal{N} \setminus \mathcal{C}) \cap S) + e(\mathcal{C}, \mathcal{N} \setminus \mathcal{C}) - e((\mathcal{N} \setminus \mathcal{C}) \cap T, t^*) \\
&= \sum_{s\in(\mathcal{N}\setminus\mathcal{C})\cap S} a(s) + e(\mathcal{C}, \mathcal{N} \setminus \mathcal{C}) - \sum_{t\in(\mathcal{N}\setminus\mathcal{C})\cap T} b(t)
\end{aligned}$$

[29] We write $\mathcal{N}^* \setminus \mathcal{C}^*$ instead of $(\mathcal{C}^*)^c$ to clarify the underling space is $\mathcal{N}^*$ not $\mathcal{N}$.

$$= \sum_{s\in(\mathcal{N}\setminus\mathcal{C})\cap S} a(s) + \left(\sum_{(D,E)\in\mathcal{C}\times(\mathcal{N}\setminus\mathcal{C})} u(D,E) - \sum_{(D,E)\in(\mathcal{N}\setminus\mathcal{C})\times\mathcal{C}} l(D,E) \right) - \sum_{t\in(\mathcal{N}\setminus\mathcal{C})\cap T} b(t),$$

which is nonnegative by (32). Thus $e(\mathcal{C}^*, \mathcal{N}^* \setminus \mathcal{C}^*) \geq e(\mathcal{N}^* \setminus \{t^*\}, t^*)$ for any cut $(\mathcal{C}^*, \mathcal{N}^* \setminus \mathcal{C}^*)$ separating s^* and t^* if and only (32) holds for any $\mathcal{C} \subseteq \mathcal{N}$.

It follows from the maximum-flow theorem with lower bounds (Theorem 6.1 Ahuja, Magnanti, and Orlin (1988)) that (32) implies the existence of a flow r^* from s^* to t^* that saturates all arcs of (T, t^*), that is, $r^*(t, t^*) = b(t)$ for all $t \in T$. Since $\sum_{s\in S} a(s) = 1 = \sum_{t\in T} b(t)$, we must have $r^*(s^*, s) = a(s)$ for all $s \in S$. These equalities imply that $r^*(S, \mathcal{N}) = 1$ and $r^*(\mathcal{N}, T) = 1$ and $r^*(D, \mathcal{N}) = r^*(\mathcal{N}, D)$ for all $D \in \mathcal{N} \setminus (T \cup S)$. Now define r as a restriction of r^* on $(\mathcal{N}, \mathcal{A})$. Then r satisfies all desired conditions. $\square$

By the theorem, we obtain the lemma by letting both T and S singletons.

F.3 Proof of Lemma C.1

For any (D, x) such that $x \in D \in 2^X$, define $\rho(D, x) = \sum_{E\supseteq D} r(E \setminus x, E)$. By (iii), we have $\rho(D, x) \geq 0$ for all (D, x) such that $x \in D \in 2^X$. Fix any D to show $\sum_{x\in D} \rho(D, x) = 1$. Then we have $\sum_{x\in D} \rho(D, x) = \sum_{x\in D} \sum_{E\supseteq D} r(E \setminus x, E) = \sum_{y\in D} r(D \setminus y, D) + \sum_{x\in D} \sum_{\substack{E\supseteq D \\ |E|\geq|D|+1}} r(E \setminus x, E) = \sum_{\substack{E\supseteq D \\ |E|=|D|+1}} r(D, E) + \sum_{x\in D} \sum_{\substack{E\supseteq D \\ |E|\geq|D|+1}} r(E \setminus x, E) = \sum_{\substack{E\supseteq D \\ |E|=|D|+1}} \sum_{y\in E\cap D^c} r(E \setminus y, E) + \sum_{x\in D} \sum_{\substack{E\supseteq D \\ |E|=|D|+1}} r(E \setminus x, E) + \sum_{x\in D} \sum_{\substack{E\supseteq D \\ |E|\geq|D|+2}} r(E \setminus x, E) = \sum_{y\in E} \sum_{\substack{E\supseteq D \\ |E|=|D|+1}} r(E \setminus y, E) + \sum_{x\in D} \sum_{\substack{E\supseteq D \\ |E|\geq|D|+2}} r(E \setminus x, E) = \sum_{x\in E} \sum_{\substack{E\supseteq D \\ |E|\geq|D|+1}} r(E \setminus x, E)$, where the third equality holds by appling (ii) for the first term; the fourth equality is obtained by rewriting the first term and dividing the second term into the two terms; and the second to the last equality is obtained by combining the first two terms into one. Note that the last term has the same form as the term in the first equation but in the last term the summation over $E = D$ is deleted. By repeating this, we get $\sum_{x\in D} \rho(D, x) = \sum_{x\in E} \sum_{\substack{E\supseteq D \\ |E|\geq|D|+2}} r(E \setminus x, E)$. Finally we get $\sum_{x\in D} \rho(D, x) = \sum_{y\in X} r(X \setminus y, X)$, which is equal to 1 by (i).

F.4 Proof of Claim C.3

We first construct a directed path from $\emptyset$ to $\tilde{X}$ that avoids every node in $\hat{\mathcal{D}}$ by adding the elements of $\tilde{X}$ one at a time. Note that any intermediate node A on this path does not appear in an essential test collection: the only test collection that contains A is $\{A \cup E : E \in 2^{X^*}\}$, which is nonessential.

In the same manner, we construct a directed path from $\tilde{X}$ to X that avoids every node in $\mathcal{H}$ by adding the elements of X^* one at a time. Any intermediate node on this path has the form $\tilde{X} \cup E$ for some $E \subseteq X^*$ (i.e., $E \in 2^{X^*}$). Such a node cannot belong to any essential test collection: any test collection that contains it must be of the form $\{\tilde{X} \cup E : E \in \mathcal{E}\}$, which is not essential.

By combining these two directed paths, we obtain a desirable $\emptyset - X$ directed path avoiding any nodes in $\mathcal{H}$.

F.5 Proof of Claim C.4

Fix $D \in \mathcal{C}^*$. Consider

- an $\emptyset - D$ directed path Π_2 containing going through the node $D \setminus A^*$,
- an $A^* - X$ directed path Π_3 which avoid any nodes in $\mathcal{H}$ (Such a path exists because we can take the union of any $A^* - \tilde{X}$ directed path and any $\tilde{X} - X$ directed path as in Claim C.3.),
- The directed path Π_4 from A^* to D which follows the same order as Π_3.

Fix $\varepsilon > 0$ and define $r^1_D \equiv (1-\varepsilon) r^{\Pi_1} + \varepsilon r^{\Pi_2} + \varepsilon r^{\Pi_3} - \varepsilon r^{\Pi_4}$. Note that r^1_D satisfies the three conditions in Lemma C.1. To confirm the condition (iii) is satisfied it suffices to show that all negative flows are cancelled in the sum $\sum_{E:E \supseteq D} r(E \setminus x, E)$. (All of the negative flows are in Π_4 and are canceled by some flow in Π_3 because Π_4 follows the same order as Π_3.)

Note also that in the flow, $\delta_{r^1_D}(\{A^*\}) = \delta_{r^1_D}(\{D \setminus A^*\}) = \varepsilon$ and $\delta_{r^1_D}(\{\tilde{X}\}) = -1$. To see $\delta_{r^1_D}(\{D\}) = -\varepsilon$ note that an arc going into D exists and is observable because A^* is not empty and consists of observable alternatives.

Moreover, for all other $H \in \mathcal{H}$, $\delta_{r^1_D}(\{H\}) = 0$. (To see this note that δ is non-zero only when observable inflows are not equal to observable outflows.[30])

Since $A^*, \tilde{X}, D \setminus A^* \notin \mathcal{H}$ by Lemma B.7, we have $\delta(\{H\}) = 0$ for any $H \in \mathcal{H} \setminus D$. This completes the proof of the claim.

F.6 Proof of Claim C.5

Choose any directed path from $\emptyset - X$ that goes through nodes A, D, and $\tilde{X} \cup D$. We denote the path by Π_5. Define $r^2_D \equiv (1-\varepsilon)r^{\Pi_1} + \varepsilon r^{\Pi_5}$. Note that r^2_D satisfies the all conditions in Lemma C.1.

In the flow, $\delta_{r^2_D}(\{A^*\}) = \delta_{r^2_D}(\tilde{X} \cup D) = -\varepsilon$ and $\delta_{r^2_D}(\{D\}) = \varepsilon$.

Moreover, for all other $H \in \mathcal{H}$, $\delta_{r^2_D}(\{H\}) = 0$ by the same reasoning as the previous claim. Since $A^*, \tilde{X} \cup D \notin \mathcal{H}$, we have $\delta_{r^2_D}(\{H\}) = 0$ for all $H \in \mathcal{H} \setminus D$. This completes the proof of the claim.

F.7 Details of Proof of Statement (ii) in Lemma C.2

We first prove a new claim:

Claim F.2. *Fix $\varepsilon \in (0,1]$. There exists a flow $\hat{r}$ such that (i) $-1/(2|\mathcal{C}^*|) = \delta_{\hat{r}}(\mathcal{C}^*) \leq \delta_{\hat{r}}(\mathcal{C})$ for any essential test collection $\mathcal{C}$; (ii) If $\mathcal{C} \not\supseteq \mathcal{C}^*$ then $\delta(\mathcal{C}) \geq 0$.*

Proof. Define $r^3 \equiv \sum_{D \in \mathcal{C}^*} \frac{1}{|\mathcal{C}^*|} r^1_D$.Then, $\delta_{r^3}(\{D\}) = -\frac{\varepsilon}{|\mathcal{C}^*|}$ for each $D \in \mathcal{C}^*$; moreover, for all $\hat{D} \in \hat{\mathcal{D}} \setminus \mathcal{C}^*$, $\delta_{r^3}(\{\hat{D}\}) = 0$. Define $r^4 \equiv \frac{1}{|\mathcal{C}^*|} r^{\Pi_1} + \frac{|\mathcal{C}^*|-1}{|\mathcal{C}^*|} r^2_{A^* \cup X^*}$. Then, $\delta_{r^4}(\{A^* \cup X^*\}) = \frac{|\mathcal{C}^*|-1}{|\mathcal{C}^*|}\varepsilon$; moreover, for all $\hat{D} \in \hat{\mathcal{D}} \setminus \mathcal{C}^*$, $\delta_{r^4}(\{\hat{D}\}) = 0.$[31] Define $\hat{r} = 1/2r^3 + 1/2r^4$. In $\hat{r}$, we have $\delta_{\hat{r}}(\mathcal{C}^*) = \sum_{D \in \mathcal{C}^*} \delta_{\hat{r}}(\{D\}) = \frac{1}{2}\big(-1 + \frac{|\mathcal{C}^*|-1}{|\mathcal{C}^*|}\big)\varepsilon = -\frac{1}{2|\mathcal{C}^*|}\varepsilon$.

Step 1: $\delta_{\hat{r}}(\mathcal{C}^*) \leq \delta_{\hat{r}}(\mathcal{C})$ for any essential test collection $\mathcal{C}$.

Proof. Fix any essential test collection $\mathcal{C}$. We consider the following two cases.

Case 1: There exists $D \in \mathcal{C}$ such that $D \setminus X^* \neq A^*$. (In fact, in this case, by the definition of essential test collection, $D \setminus X^* \neq A^*$ for all $D \in \mathcal{C}$.) Then, $\mathcal{C} \cap \mathcal{C}^* = \emptyset$. Since $\hat{D} \in \hat{\mathcal{D}} \setminus \mathcal{C}^*$, $\delta_{\hat{r}}(\{\hat{D}\}) = 0$, we have $\delta_{\hat{r}}(\mathcal{C}) = 0 \geq \delta(\mathcal{C}^*)$.

[30] In the figure, this occurs when a dotted line becomes a solid line or vice-versa in the diagram.
[31] Note that $A^* \cup X^* \in \mathcal{C}^*$.

Case 2: $D \setminus X^* = A^*$ for all $D \in \mathcal{C}$. Since $\mathcal{C}$ is complete, $\mathcal{C}$ contains $A^* \cup X^*$. Since $\mathcal{C}^*$ contains all $D \in \hat{\mathcal{D}}$ such that $\delta_{\hat{r}}(\{D\}) < 0$ it is clear that $\delta_{\hat{r}}(\mathcal{C}) \geq \delta_{\hat{r}}(\mathcal{C}^*)$. ■

Step 2: If $\mathcal{C} \not\supseteq \mathcal{C}^*$ then $\delta(\mathcal{C}) \geq 0$.

Proof. Suppose that $\mathcal{C} \not\supseteq \mathcal{C}^*$. Then there exists $D^* \in \mathcal{C}^*$ such that $D^* \notin \mathcal{C}$. By Step 1, $\delta_{\hat{r}}(\mathcal{C} \cup \{D^*\}) \geq \delta(\mathcal{C}^*) = -\frac{1}{2|\mathcal{C}^*|}\varepsilon$. Also by definition of $\hat{r}$, $\delta_{\hat{r}}(\{D^*\}) = -\frac{1}{2|\mathcal{C}^*|}\varepsilon$ so $\delta_{\hat{r}}(\mathcal{C}) = \delta_{\hat{r}}(\mathcal{C} \cup \{D^*\}) - \delta_{\hat{r}}(\{D^*\}) \geq 0$. ■

□

We finally prove the statement of the lemma:

Claim F.3. *There exists a flow r^* from $\emptyset$ to X such that $\delta_{r^*}(\mathcal{C}^*) < 0$ and $\delta_{r^*}(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$. Furthermore, r^* satisfies conditions (i)–(iii) in Lemma C.1 and $r(D \setminus x, D) \geq 0$ for all $x \in \tilde{X}$ and $D \supseteq x$.*

Proof. For each essential test collection $\mathcal{C}$ such that $\mathcal{C}^* \not\supseteq \mathcal{C}$, choose $D_{\mathcal{C}} \in \mathcal{C} \setminus \mathcal{C}^*$. Let $\mathcal{F}$ be the collection of such $D_{\mathcal{C}}$. Define $r^* \equiv \alpha\hat{r} + (1-\alpha)\sum_{D_{\mathcal{C}} \in \mathcal{F}} \frac{1}{|\mathcal{F}|} r^2_{D_{\mathcal{C}}}$. Then for any essential test collection $\mathcal{C}$, $\delta_{r^*}(\mathcal{C}) = \alpha\delta_{\hat{r}}(\mathcal{C}) + (1-\alpha)\sum_{D_{\mathcal{C}} \in \mathcal{F}} \frac{1}{|\mathcal{F}|}\delta_{r^2_{D_{\mathcal{C}}}}(\mathcal{C})$. Since $\delta_{r^2_{D_{\mathcal{C}}}}(D_{\mathcal{C}}) > 0$, there exists α small enough such that for any essential test collection $\mathcal{C}$ such that $\mathcal{C}^* \not\supseteq \mathcal{C}$, we have $\delta_{r^*}(\mathcal{C}) \geq 0$.

Note that $\delta_{r^*}(\mathcal{C}^*) = -\frac{\alpha}{2|\mathcal{C}^*|}\varepsilon$. Note also that $r^2_{D_{\mathcal{C}}}$ does not decrease values of δ for any test collection. Thus by statement (ii) of the previous claim, we have if $\mathcal{C} \not\supseteq \mathcal{C}^*$ then $\delta_{r^*}(\mathcal{C}) \geq 0$. It follows that $\delta_{r^*}(\mathcal{C}^*) < 0$ and $\delta_{r^*}(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$. ■

G Supplemental Contents

G.1 Interpretation of Condition (ii) in Theorem 3.2

Suppose that an incomplete dataset ρ is RU-rationalizable by μ. To understand the interpretation of condition (ii) in Theorem 3.2, fix a test collection $\mathcal{C} = \{A \cup E \mid E \in \mathcal{E}\}$ and define $\hat{\mathcal{C}} = \{C \in \mathcal{C} \mid C \setminus x^* \notin \mathcal{C}$ for some $x^* \in X^*\}$. [32] Also, for each

[32] In a Boolean lattice that we will explain later, the subcollection $\hat{\mathcal{C}}$ can be interpreted as bottom parts of $\mathcal{C}$.

$C \in \hat{\mathcal{C}}$, let $U_C = \{x^* \in X^* \mid C \setminus x^* \notin \mathcal{C}\}$.[33] We can show that the left hand side of (2) is

$$\sum_{C \in \hat{\mathcal{C}}} \mu\left(\succ \in \mathcal{L} \mid C^c \succ C \text{ and } \max_C \succ \in U_C\right), \tag{33}$$

where $\max_C \succ$ denotes the best element in C with respect to $\succ$. In particular, when the test collection $\mathcal{C}$ is a singleton of the form $\{D\}$ where $X^* \subseteq D$, (33) simplifies to $\mu(\succ \in \mathcal{L} | \exists x^* \in X^* \ s.t. \ D^c \succ x^* \succ D \setminus x^*)$.

Equation (33) can be derived as follows. By the inflow-outflow equality, the total flow out of $\mathcal{C}$ minus the total flow into $\mathcal{C}$ is zero. Note that since $\mathcal{C}$ is an essential test collection, there are no unobservable flows out of $\mathcal{C}$. Thus the left hand side of (2) is only missing the unobservable flows into $\mathcal{C}$. That is,

$$\begin{aligned}\left(\sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D\cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\in\tilde{X}} K(\rho, F\cup y, y)\right) \\ - \sum_{(F,z):F\notin\mathcal{C},F\cup z\in\mathcal{C},z\in X^*} K(\rho, F\cup z, z) = 0.\end{aligned}$$

It follows that the left hand side of (2) equals

$$\sum_{(F,z):F\notin\mathcal{C},F\cup z\in\mathcal{C},z\in X^*} K(\rho, F\cup z, z).$$

Applying equation (1) yields (33), as desired. If $\mathcal{C}$ is a singleton as in Remark 3.3, then it is of the form $C = \{\mathcal{D}\}$ where $D = A \cup X^*$ for some $\emptyset \subsetneq A \subsetneq \tilde{X}$. Then, equation (33) simplifies to equation (3).

G.2 Random Utility Polytope

In this section, we provide a geometric intuition for the set of RU rationalizable stochastic choice functions. Let $\mathcal{M}$ be the set of pairs of (D,x) such that $\rho(D,x)$

[33] In a Boolean lattice, U_C is the set of unobservable alternatives x^* where $(C \setminus x^*, C)$ is an arc from outside of $\mathcal{C}$ going into $\mathcal{C}$.

is observable.

Remark G.1. *For each ranking $\succ \in \mathcal{L}$ and $(D, x) \in \mathcal{M}$, define*

$$\rho^{\succ}(D, x) = \begin{cases} 1 & \text{if } x \succ y \text{ for all } y \in D \setminus x; \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

The stochastic choice function $\rho^{\succ}$ gives probability one to the best alternative x in a choice set D according to the ranking $\succ$. The set of RU-rationalizable datasets is a polytope, that is, co.$\{\rho^{\succ} \mid \succ \in \mathcal{L}\}$, where co. denotes the convex hull.

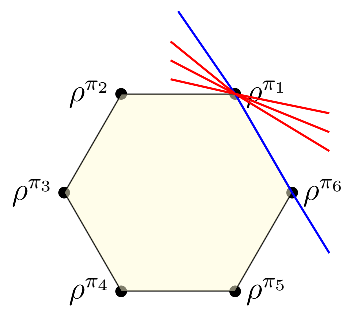

Figure 7: Random utility polytope

The hexagons in Figure 7 illustrates the polytope.[34] The inequality conditions in Theorem 3.2 consist of the facet defining inequalities of the polytope, which correspond to the blue hyperplanes. On the other hand, as we will explain in the next section, McFadden and Richter's approach would contain non-facet defining inequalities, which correspond to the red hyperplanes.

G.3 Generalized Axiom of Revealed Stochastic Preference (GARSP)

In this section we discuss two different axioms and their application to our setup. The first, which we refer to as Axiom of Revealed Stochastic Preference (ARSP),

[34]Although the geometric intuition is useful, it is important to notice that the figure oversimplifies the reality since the number (i.e., $|X|!$) of vertices and the dimension of a random utility function can be very large. To see why the dimension of a random utility function can be very large, notice that it assigns a number for each pair of $(D, x) \in \mathcal{M}$.

appears commonly throughout the literature in contexts with unobservability of menus but not unobservability of alternatives. This formulation of the axiom is necessary and sufficient when, if the analyst observes the choice frequency of an alternative from one menu, all choice frequencies from that menu are observable. It may fail, however, when some alternatives are unobservable. The second axiom, which we call GARSP, is necessary and sufficient in our setup with unobservable alternatives. An equivalent formulation of GARSP may be found in McFadden and Richter (1990). Unlike our characterization (Theorem 3.2) both of these axioms contain an infinite number of redundant inequalities. We further investigate the relationship between the Theorem 3.2 and GARSP in the next section.

Let $\mathcal{M}$ be the set of pairs (D, x) such that $x \in D \in 2^X$ and $\rho(D, x)$ is well defined (i.e., $\rho(D, x)$ is observable to the analyst). In the following, we call $\mathcal{M}$ the set of observable pairs. We write the set of datasets as $\mathcal{P}(\mathcal{M})$. In McFadden and Richter (1990), it is assumed that all alternatives are observable. That is, if $(D, x) \in \mathcal{M}$ then $(D, y) \in \mathcal{M}$ for all $y \in D$. Equivalently, $\mathcal{M} \coloneqq \{(D, x) \in \mathcal{D} \times X \mid x \in D\}$ for some $\mathcal{D} \subseteq 2^X \setminus \emptyset$.[35] In the main body of the paper, we assumed $\mathcal{M} \equiv \{(D, x) \in \mathcal{D} \times \tilde{X} \mid x \in D\}$.[36] In the following section we will consider arbitrary $\mathcal{M} \subseteq \{(D, x) \mid x \in D \in 2^X\}$.

Definition G.2. *A stochastic choice function satisfies ARSP if for any sequence finite $(D_i, x_i)_{i=1}^k \in \mathcal{M}^k$ for all k,*

$$\max_{\succ \in \mathcal{L}(X)} \sum_i \rho^{\succ}(D_i, x_i) \geq \sum_i \rho(D_i, x_i).$$

This is necessary and sufficient when there are no unobservable alternatives. However, this is not sufficient to characterize random utility in our setup. Consider the following example:

Remark G.3. *Let $X = \{a, b, c, d\}$ and $X^* = \{c, d\}$ and $\mathcal{M} = \{(X, a), (X \setminus b, a)\}$. Let ρ be such that $\rho(X, a) = 0$ and $\rho(X \setminus b, a) = 1$. This is obviously not RU-rational*

[35]The formal definition of $\mathcal{P}(\mathcal{M})$ can be provided as follows: $\mathcal{P}(\mathcal{M}) = \{\rho \in \mathbf{R}_+^{\mathcal{M}} \mid$ For all $D \in \mathcal{D}$ (i) if $(D, x) \in \mathcal{M}$ for any $x \in D$, then $\sum_{x \in D} \rho(D, x) = 1$; (ii) if $(D, x) \notin \mathcal{M}$ for some $x \in D$, then $\sum_{x \in D} \rho(D, x) \leq 1\}$.

[36]In Falmagne (1978), $\mathcal{M}$ is assumed to be $\{(D, x) \mid x \in D \in 2^X\}$.

as it violates monotonicity (in particular condition (i) is violated). However, fix $\succ_0 \in \mathcal{L}(X)$ be such that $a \succ_0 b \succ_0 c \succ_0 d$. That is, $\rho^{\succ_0}(X, a) = \rho^{\succ_0}(X \setminus b, a) = 1$. Then since $\rho^{\succ_0}(D, x) \geq \rho(D, x)$ for all $(D, x) \in \mathcal{M}$, we observe that

$$\sum_i \rho^{\succ_0}(D_i, x_i) \geq \sum_i \rho(D_i, x_i)$$

and thus since $\max_{\succ \in \mathcal{L}(X)} \sum_i \rho^{\succ}(D_i, x_i) \geq \sum_i \rho^{\succ_0}(D_i, x_i)$, ρ satisfies ARSP but not RU rationalizability.

We provide the following characterization that works for arbitrary $\mathcal{M} \subseteq \{(D, x) \in 2^X \times X \mid x \in D\}$. We call it the *Generalized Axiom of Revealed Stochastic Preference (GARSP)*

Definition G.4. *A stochastic choice function ρ satisfies GARSP axiom if for any finite sequences $(D_i^+, x_i^+)_{i=1}^k \in \mathcal{M}^k$ and $(D_j^-, x_j^-)_{i=1}^l \in \mathcal{M}^l$ for all k and l,*

$$\max_{\succ \in \mathcal{L}(X)} \sum_i \rho^{\succ}(D_i^+, x_i^+) - \sum_j \rho^{\succ}(D_j^-, x_j^-) \geq \sum_i \rho(D_i^+, x_i^+) - \sum_j \rho(D_j^-, x_j^-).$$

Theorem G.5. *Suppose $\mathcal{M} \subseteq \{(D, x) \in 2^X \times X \mid x \in D\}$. A stochastic choice function ρ defined on $\mathcal{M}$ is RU-rationalizable if and only if it satisfies GARSP.*

Proof. First suppose that there exists sequences $(D_i^+, x_i^+)_{i=1}^k \in \mathcal{M}^k$ and $(D_j^-, x_j^-)_{i=1}^l \in \mathcal{M}^l$ for all k and l such that $\max_{\succ \in \mathcal{L}(X)} \sum_i \rho^{\succ}(D_i^+, x_i^+) - \sum_j \rho^{\succ}(D_j^-, x_j^-) < \sum_i \rho(D_i^+, x_i^+) - \sum_j \rho(D_j^-, x_j^-)$. Then for any $\rho^\mu \in \mathcal{P}_r = \mathrm{co}\left(\{\rho^{\succ} \mid \succ \in \mathcal{L}\}\right)$ it follows that $\sum_i \rho^\mu(D_i^+, x_i^+) - \sum_j \rho^\mu(D_j^-, x_j^-) < \sum_i \rho(D_i^+, x_i^+) - \sum_j \rho(D_j^-, x_j^-)$. We conclude that $\rho \notin \mathcal{P}_r$.

To show the other direction, suppose that $\rho \notin \mathcal{P}_r$. Since $\mathcal{P}_r$ is compact and convex then by the separating hyperplane theorem there exists $\nu : \mathcal{M} \to \mathbb{R}$ such that $\nu \cdot \rho^\mu < \nu \cdot \rho$ for all $\rho^\mu \in \mathcal{P}_r$. In particular,

$$\max_{\succ \in \mathcal{L}(X)} \sum_{(D,x) \in \mathcal{M}} \nu(D, x) \rho^{\succ}(D, x) < \sum_{(D,x) \in \mathcal{M}} \nu(D, x) \rho(D, x). \tag{35}$$

Now by density of the rationals and since the maximum is over a finite set, ν can be taken to be rational valued. Then by multiplying by a large positive integer, ν

can also be taken to be integer valued and still satisfy the inequality. Finally, we obtain the result by letting (D, x) appear $\nu(D, x)$ times in $(D_i^+, x_i^+)_{i=1}^k \in \mathcal{M}^k$ if it is positive and $\nu(D, x)$ times in $(D_i^-, x_i^-)_{j=1}^l \in \mathcal{M}^l$ if it is negative. $\square$

This characterization is similar to ARSP, but it allows negative signs in the sum. The fact that we need negative signs is clear geometrically. As Figure 7 shows, we are characterizing the polytope by considering *all* hyperplanes, equivalently *all* normal vectors. Thus, we should consider the negative directions as well as the positive directions.

In McFadden and Richter (1990), the observability condition on choice frequencies from each menu is that we observe a probability distribution over a weak field of subsets of the menu. In our setup, this is equivalent to the the following condition: whenever we observe the probability $\rho(D, x)$ that x is chosen from D, that we also observe the chance that x is not chosen from D, denoted by $\rho(D, D\backslash x)$, which we can obtain by $1 - \rho(D, x)$.

Based on the assumption, we replace all $-\rho(D_j^-, x_j^-)$ (respectively $-\rho^{\succ}(D_j^-, x_j^-)$) with $\rho(D_j^-, D_j^- \setminus x_j^-) \equiv 1 - \rho(D_j^-, x_j^-)$ (respectively $\rho^{\succ}(D_j^-, D_j^- \setminus x_j^-) \equiv 1 - \rho^{\succ}(D_j^-, x_j^-)$) in the axiom. Notice that this operation keeps the equality because we are simply adding one l-times to the both hand sides. This rewriting gives us the original axiom by McFadden and Richter (1990) (page 171)[37]: for any sequence of pairs (D_i, A_i) such that $A_i \subseteq D_i$ and $\rho(D_i, A_i)$ is observable for all i,

$$\max_{\succ \in \mathcal{L}(X)} \sum_i \rho^{\succ}(D_i, A_i) \geq \sum_i \rho(D_i, A_i). \tag{36}$$

Note that $\rho(D_i, A_i)$ is observable in our setup if and only if $D_i \in \mathcal{D}$ and $[A_i \subseteq \tilde{X}$ or $A_i^c \subseteq \tilde{X}]$.

If the dataset satisfies the following condition, then ARSP becomes equivalent to GARSP. In other words, we can focus on positive sequences: For each $D \in \mathcal{D}$,

[37] McFadden and Richter (1990) call their original axiom as ARSP, not GARSP.

we can observe $\rho(D,x)$ for all $x \in D$. Thus,

$$\sum_{x \in D \text{ s.t.} (D,x) \in \mathcal{M}} \rho(D,x) = 1. \tag{37}$$

Theorem G.6. *Let $\mathcal{M} \subseteq \{(D,x) \in 2^X \times X \mid x \in D\}$. Suppose that if $(D,x) \in \mathcal{M}$, then $(D,y) \in \mathcal{M}$ for all $y \in D$. A stochastic choice function ρ defined on $\mathcal{M}$ is RU-rationalizable if and only if it satisfies ARSP.*

The necessity of the proof does not change. In the sufficiency part of proof above, we showed the existence of ν satisfying (35). Let $s = -\min_{(D,x) \in \mathcal{M}} \nu(D,x)$. Then define v^* by $v^*(D,x) = v(D,x) + s$ for all $(D,x) \in \mathcal{M}$. Note that ν^* is a nonnegative vector and for any stochastic choice function for each $D \in \mathcal{D}$, $\sum_{x \in D \ s.t. \ (D,x) \in \mathcal{M}} \nu^*(D,x)\rho(D,x) - \left(\sum_{x \in D \ s.t. \ (D,x) \in \mathcal{M}} \nu(D,x)\rho(D,x) \right) = s$, where the last equality holds by (37). Thus, (35) holds with ν^* in the place of ν. The rest of the proof is the same.

One drawback that this characterization shares with McFadden and Richter (1990) but not the characterization in Theorem 3.2 is that it has an infinite number of inquealities to test, some of them being redundant. In the following we show that these redundancies are inherent to the approach.

G.4 Redundancy in the McFadden and Richter (1990) Approach

The conditions of McFadden and Richter (1990) and our generalization involve an infinite number of sequences and some of the conditions are redundant. In this section, we further clarify the relationship between our approach and the approach taken by McFadden and Richter (1990). The main message is that the approach by McFadden and Richter (1990) contain redundancy in an essential way, unlike the BM polynomials. [38]

[38] In other words, if one removes all redundancy from the results by McFadden and Richter (1990), such results should reduce to Falmagne (1978) for the case of complete datasets and our results for the case of incomplete datasets.

Notice in Definition G.4, the same $(D,x) \in \mathcal{M}$ may appear arbitrarily many times in the sequences $(D_i^+, x_i^+)_{i=1}^k \in \mathcal{M}^k$ and $(D_j^-, x_j^-)_{i=1}^l \in \mathcal{M}^l$. Thus, the number of sequences to be tested in the GARSP is infinite, although there are finitely many pairs $(D,x) \in \mathcal{M}$. McFadden and Richter (1990) discuss the difficulty of providing an upper bound on the number of allowable repetitions needed for their axiom to fully characterize RUM. They prove that sequences containing repetitions must be tested in general. In the following section, we show how Theorem 3.2 can provide an upper bound on the number of required repetitions. We then show that while limiting the number of repetitions does make the number of inequalities to test finite, the number of inequalities to test is much larger than our independent conditions obtained in Theorem 3.2 and therefore contains a large amount of redundancy.

We first show the following remark

Remark G.7. *The inequality conditions (i) of Theorem 3.2 can be written as the GARSP with no repetitions.*

For each (E,y) such that $y \in E$, define $K_{(E,y)} \in \mathbf{R}^{\mathcal{M}}$ by

$$K_{(E,y)}(D,x) = \begin{cases} -1 & \text{if } y = x \text{ and } D \supseteq E \text{ and } |D \setminus E| \text{ is even,} \\ +1 & \text{if } y = x \text{ and } D \supseteq E \text{ and } |D \setminus E| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

For any $\rho \in \mathcal{P}$, we have $K_{(E,y)} \cdot \rho = -K(\rho, E, y)$. For each (E,y), define sequences $(D_i^+, x_i^+)_{i=1}^k \in \mathcal{M}^k$ and $(D_j^-, x_j^-)_{i=1}^l \in \mathcal{M}^l$ so that each (D,x) appears exactly once in $(D_i^+, x_i^+)_{i=1}^k$ if $K_{(E,y)}(D,x) = 1$ and exactly once in $(D_j^-, x_j^-)_{i=1}^l \in \mathcal{M}^l$ if $K_{(E,y)}(D,x) = -1$. Thus $\sum_i \rho(D_i^+, x_i^+) - \sum_j \rho(D_j^-, x_j^-) = -K(\rho, E, y)$ for all ρ.

Notice then that

$$\max_{\succ \in \mathcal{L}(X)} \sum_i \rho^{\succ}(D_i^+, x_i^+) - \sum_j \rho^{\succ}(D_j^-, x_j^-) = \max_{\succ \in \mathcal{L}(X)} \sum_{(D,x) \in \mathcal{M}} -K(\rho^{\succ}, E, y) = 0.$$

GARSP for this sequence is equivalent to $K(\rho, E, y) \geq 0$. In particular, in the case of complete data, it is sufficient to only consider sequences without repetition.

We can do something similar with condition (ii) in Theorem 3.2.

Remark G.8. *The inequality conditions (ii) of Theorem 3.2 are also implied by GARSP.*

For each essential test collection $\mathcal{C}$ define $\nu_{\mathcal{C}}(D,x)$ as the coefficient on $\rho(D,x)$ in the polynomial below:

$$\delta'_\rho(\mathcal{C}) \equiv \sum_{\substack{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C},\\(D\cup x,x)\in\mathcal{M}}} K(\rho,D\cup x,x) - \sum_{\substack{(E,y):E\notin\mathcal{C},E\cup y\in\mathcal{C},\\(E\cup y,y)\in\mathcal{M}}} K(\rho,E\cup y,y). \tag{38}$$

Note that (38) is equivalent to the left hand-side of condition (ii) of Theorem 3.2. The value of (38) coincided with $\delta_\rho(\mathcal{C})$ defined in (16) when $\mathcal{C}$ is an essential test collection.[39]

Define sequences $(D_i^+,x_i^+)_{i=1}^k \in \mathcal{M}^k$ and $(D_j^-,x_j^-)_{i=1}^l \in \mathcal{M}^l$ so that each (D,x) appears exactly $-\nu_{\mathcal{C}}(D,x)$ in $(D_i^+,x_i^+)_{i=1}^k$ if $\nu_{\mathcal{C}}(D,x) < 0$ and exactly $\nu_{\mathcal{C}}(D,x)$ many times in $(D_j^-,x_j^-)_{i=1}^l \in \mathcal{M}^l$ if $\nu_{\mathcal{C}}(D,x) > 0$. Then $\sum_i \rho(D_i^+,x_i^+) - \sum_j \rho(D_j^-,x_j^-) = -\delta'_\rho(\mathcal{C})$ for all ρ. As shown in the proof of Theorem 3.2, $\delta'_{\rho^\succ}(\mathcal{C}) \geq 0$ for all $\succ$; and furthermore there exists $\succ$ such that $\delta'_{\rho^\succ}(\mathcal{C}) = 0$ by the fact that $\delta'_\rho(\mathcal{C}) = 0$ is a facet defining inequality and the vertices of the polytope consist of $\rho^\succ$s. Thus

$$\max_{\succ\in\mathcal{L}(X)} \sum_i \rho^\succ(D_i^+,x_i^+) - \sum_j \rho^\succ(D_j^-,x_j^-) = \max_{\succ\in\mathcal{L}(X)} \sum_{(D,x)\in\mathcal{M}} -\delta'_{\rho^\succ}(\mathcal{C}) = 0$$

and therefore GARSP for this sequence is equivalent to condition (ii) for the essential test collection $\mathcal{C}$.

Remark G.9. *We have thus shown our conditions can be written as GARSP (by selecting sequences in a very particular way); however the converse is not true. While our conditions are non-redundant, all of the sequences in GARSP apart from those corresponding to our conditions are redundant.*

[39] When $\mathcal{C}$ is an essential test collection, we have $X \notin \mathcal{C}$ and $\emptyset \notin \mathcal{C}$.

See Figure 7 for the illustration. Our conditions contain only the blue facet defining hyperplanes, while the GARSP contain all non-facet defining red hyperplanes.

To get an idea of how much redundancy there is in GARSP, consider the following counting argument. Suppose $\mathcal{D} = 2^X \setminus \emptyset$. For finite sequences in $\mathcal{M}$ with no repetitions, each element of $\mathcal{M}$ either appears as a positive, negative, or does not appear in the sequence. Thus (up to reordering) there are $3^{|\mathcal{M}|}$ inequalities without repetitions. This is at least $2^{2^{|X|-1}}$ as the lower bound of $|\mathcal{M}|$ is $2^{|X|}-1$. Note that this is a lower bound for the number of sequences *without* repetitions. Since repetitions are necessary when the datasets are incomplete, this lower bound is very small one.

We now obtain an upper bound for the number of inequalities in condition (ii) of Theorem 3.2. The number of inequalities in condition (i) of Theorem 3.2 is $|\mathcal{M}|$, which is bounded by $2^{|X|} \times |\tilde{X}|$. To get an upper bound for the number of inequalities in condition (ii) of Theorem 3.2, notice that the number of essential test collections is less than $2^{|\tilde{X}|} 2^{2^{|X^*|}} = 2^{2^{|X^*|}+|\tilde{X}|}$ since there are less than $2^{|\tilde{X}|}$ options for A and less than $2^{2^{|X^*|}}$ options for $\mathcal{E}$. Thus, the upper bound for the number of inequalities in condition (ii) is $2^{2^{|X^*|}+|\tilde{X}|} + 2^{|X|}|\tilde{X}|$. We have

$$2^{2^{|X^*|}+|\tilde{X}|} + 2^{|X|}|\tilde{X}| = 2^{\,2^{|X^*|}+|\tilde{X}|}\Big(1 + |\tilde{X}|\, 2^{\,|X^*|-2^{|X^*|}}\Big)$$

as $|X| = |\tilde{X}| + |X^*|$. For $|X^*| \geq 2$ we have $|X^*| - 2^{|X^*|} \leq -2$, thus $2^{\,|X^*|-2^{|X^*|}} \leq \frac{1}{4}$. Hence

$$1 + |\tilde{X}|\, 2^{\,|X^*|-2^{|X^*|}} \;\leq\; 1 + \frac{|\tilde{X}|}{4} \;\leq\; 2^{\,|\tilde{X}|-1},$$

where the last inequality holds because $|\tilde{X}| \geq 2$. Thus an upper bound is

$$2^{2^{|X^*|}+|\tilde{X}|} + 2^{|X|}|\tilde{X}| = 2^{\,2^{|X^*|}+|\tilde{X}|}\Big(1 + |\tilde{X}|\, 2^{\,|X^*|-2^{|X^*|}}\Big) \leq 2^{\,2^{|X^*|}+2|\tilde{X}|-1}$$

Therefore the ratio is at least

$$\frac{2^{2^{|X|-1}}}{2^{2^{|X^*|}+2|\tilde{X}|-1}} = 2^{2^{|X|}-2^{|X^*|}-2|\tilde{X}|} \geq 2^{2^{|X|-2}}$$

for $|\tilde{X}| \geq 2$ and $|X^*| \geq 2$. Summarizing the above argument, we have the following:

Remark G.10.

$$\frac{\#(\textit{inequalities of GARSP without repetitions})}{\#(\textit{inequalities of Theorem 3.2})} \geq 2^{2^{|X|-2}}$$

for $|\tilde{X}| \geq 2$ and $|X^*| \geq 2$. This illustrates that even when restricting attention to non-repeating sequences, the number of inequalities required for GARSP is substantially larger than the number of essential test collections identified in Theorem 3.2.

The remaining question is how many repetitions are required to be sufficient in characterizing RUM. By using Theorem 3.2, we will obtain a lower bound for the number of repetitions. By the previous observation, this is a lower bound on the number of times that any given $\rho(D, x)$ appears in condition (ii) of Theorem 3.2. We first construct a test collection. Take arbitrary $A = \tilde{X} \setminus b$ for some $b \in \tilde{X}$. Now take $\mathcal{E} \subseteq 2^{X^*}$ that contains all subsets of X^* with at least $|X^*|/2$ elements. Note then that $\{A \cup E | E \in \mathcal{E}\}$ is an essential test collection.

Now take $a \in A$. We wish to count how many time $\rho(X, a)$ appears in the inequality corresponding to the test collection. First, notice $K(\rho, A \cup E, a)$ only appears as an inflow in the inequality. Now, the sign of $\rho(X, a)$ alternates in $K(\rho, A \cup E, a)$ with respect to the size of E. Thus it appears (up to a change of sign) $\sum_{E \in \mathcal{E}} (-1)^{|E|} = \sum_{k=0}^{m/2} (-1)^k \binom{m}{k}$ times. This equals $(-1)^{m/2} \binom{m-1}{m/2}$ where $m = |X^*|$. Now, it is well-known that $\binom{n}{k} \geq (n/k)^k$ for all n and k, Thus,

$$\binom{|X^*| - 1}{|X^*|/2} \geq \frac{1}{|X^*|} \binom{|X^*|}{|X^*|/2} \geq \frac{2^{|X^*|/2}}{|X^*|}.$$

Thus, we have the following:

Remark G.11. *Let $\mathcal{M} = \{(D, x) \in 2^X \times \tilde{X} | x \in D\}$. Then, in order for GARSP axiom to be sufficient for RU-rationalizability, we must test the axiom for sequences containing at least $\frac{2^{|X^*|/2}}{|X^*|}$ repetitions.*

The two remarks above demonstrate that the number of inequalities in Theorem 3.2 is significantly smaller than the number of inequities required in McFadden-Richter approach.

G.5 Simplification of bounds of unobservable choice probabilities when $\mathcal{D} = 2^X \setminus \emptyset$

In this section, we assume that $\mathcal{D} = 2^X \setminus \emptyset$, we provide further simplification of bounds of unosbservavle choice frequencies.

Corollary G.12. *Let $\mathcal{D} = 2^X \setminus \emptyset$. For $(D, x) \notin \mathcal{M}$, the upper bound is obtained by*

$$\overline{\rho}(D, x) = \max_{\{r(D \setminus x, D)\}_{(D,x) \notin \mathcal{M}}} \sum_{A': A \subseteq A' \subseteq \tilde{X}} \sum_{E': E \subseteq E' \subseteq X^*} r(A' \cup E' \setminus x, A' \cup E') \tag{39}$$

subject to

$$\sum_{y \in E'} r(A' \cup E' \setminus y, A' \cup E') - \sum_{y \in X^* \setminus E'} r(A' \cup E', A' \cup E' \cup y) = \delta_\rho(A' \cup E') \tag{40}$$

for all $A' \subseteq \tilde{X}$ and $E' \subseteq X^*$. The lower bound $\underline{\rho}(D, x)$ solves a similar problem with a min replacing the max.

Remark G.13.

- *The form of (40) implies that for $A', A'' \subseteq \tilde{X}$, $E', E'' \subseteq X^*$, and $y \in E', z \in E''$, if $A' \neq A''$, then variables $r(A' \cup E' \setminus y, A' \cup E')$ and $r(A'' \cup E'' \setminus z, A'' \cup E'')$ are independent; either of them does not restrict the other via the constraints (40).*
- *This means that each constraint can be considered separately.*
- *Therefore, we can optimize the inner sum of (39) separately. That is, the maximum value of the problem is equivalent to the sum of the maximum values of the following problems for all A' such that $A \subseteq A' \subseteq \tilde{X}$:*

$$\max_{\{r(D \setminus x, D)\}_{(D,x) \notin \mathcal{M}}} \sum_{E': E \subseteq E' \subseteq X^*} r(A' \cup E' \setminus x, A' \cup E') \tag{41}$$

subject to (40) for all $E' \subseteq X^*$.

- *A large linear program (39) is now decomposed into smaller problems (41), which improves the computational efficiency, especially when A is large.*

G.6 Implications to statistical testing of rationality

Theorem 3.2 establishes that following two representations of the set of random utility polytope are equivalent: $co.\{\rho^{\succ} \mid \succ \in \mathcal{L}\}$ and

$$\left\{\rho \in \mathbf{R}_{+}^{\mathcal{M}} \mid \rho \text{ satisfies the inequalities in (i) and (ii) of Theorem 3.2}\right\}.$$

The first representation characterizes an RU-rationalizable dataset as the convex hull of a finite set of points. In convex geometry, this is known as a V-representation. The second representation expresses the polytope via its facet defining inequalities, commonly referred to as an H-representation.

For testing whether a given incomplete dataset is RU-rationalizable, the H-representation is generally more practical, as it requires only checking for a single violated inequality to reject rationalizability. In contrast, the V-representation involves searching the typically high-dimensional space $\Delta(\mathcal{L})$ for a supporting weight, making it computationally intensive.

A key challenge with the H-representation is that a closed form expression remains unknown for arbitrary patterns of missing data, whereas the V-representation follows directly from the definition of rationalizability. Consequently, empirical studies on testing RU-rationalizability, such as Kitamura and Stoye (2018) and Dean, Ravindran, and Stoye (2022), rely on the V-representation, trading computational cost for broader applicability.

Our contribution in Theorem 3.2 is to explicitly enumerate all facet defining inequalities for a specific structure of data incompleteness. Once the H-representation is obtained, rationality testing reduces to verifying inequality constraints. Statistical inference can then proceed using standard techniques for moment inequalities (see Andrews and Soares (2010), Canay, Illanes, and Velez (2023)), though assessing the performance of such methods is beyond the scope of this paper.

G.7 Values of BM Polynomials of Example in Section 1.1

In this section, we provide a table that gives all of values of BM polynomials that can be calculated given ρ in Table 1 of Section 1.1 for $\varepsilon = 0$.

$$K(\rho, D, x) :$$

$D \backslash x$	a	b	(c)	(d)
$\{a\}$	0	–	–	–
$\{b\}$	–	0	–	–
$\{a,b\}$	0	1/3	–	–
$\{a,c\}$	1/6	–	?	–
$\{a,d\}$	1/6	–	–	?
$\{b,c\}$	–	1/6	?	–
$\{b,d\}$	–	1/6	–	?
$\{c,d\}$	–	–	?	?
$\{a,b,c\}$	1/6	0	?	–
$\{a,b,d\}$	1/6	0	–	?
$\{a,c,d\}$	1/6	–	?	?
$\{b,c,d\}$	–	1/6	?	?
$\{a,b,c,d\}$	1/6	1/6	?	?

G.8 Details of calibrated dataset

Recall that $\mathcal{L}$ is the set of linear orders on $X = \{0,1,2,3,4\}$. For a probability distribution μ over $\mathcal{L}$, let $\rho(D, x \mid \mu)$ be the choice probability of x out of D induced by μ, that is,

$$\rho(D, x \mid \mu) \coloneqq \mu\left(\{\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x\}\right).$$

Given a complete dataset ρ, we shall find an RU-rationalizable dataset that resembles ρ. We solve the following problem:

$$\hat{\mu} \in \underset{\mu \in \Delta(\mathcal{L})}{\operatorname{argmin}} \sum_{x \in D \in \mathcal{D}} \left(\rho(D, x \mid \mu) - \rho(D, x)\right)^2,$$

and we use the calibrated choice data $(\rho(D, x \mid \hat{\mu}))_{x \in D \in \mathcal{D}}$ in the analysis in Section 4. Table 2 summarizes the calibrated probabilities for the lottery dataset used in the analysis of Section 4.

Table 2: Calibrated choice data ρ

$D \backslash x$	0	1	2	3	4
{0, 1}	0.434996	0.565004	-	-	-
{0, 2}	0.171289	-	0.828711	-	-
{0, 3}	0.119875	-	-	0.880125	-
{0, 4}	0.108795	-	-	-	0.891205
{1, 2}	-	0.444959	0.555041	-	-
{1, 3}	-	0.203263	-	0.796737	-
{1, 4}	-	0.179475	-	-	0.820525
{2, 3}	-	-	0.475196	0.524804	-
{2, 4}	-	-	0.307597	-	0.692403
{3, 4}	-	-	-	0.453832	0.546168
{0, 1, 2}	0.162682	0.291693	0.545626	-	-
{0, 1, 3}	0.119875	0.156422	-	0.723703	-
{0, 1, 4}	0.091033	0.121704	-	-	0.787263
{0, 2, 3}	0.064422	-	0.431887	0.503691	-
{0, 2, 4}	0.066709	-	0.295601	-	0.63769
{0, 3, 4}	0.053548	-	-	0.453832	0.49262
{1, 2, 3}	-	0.198698	0.289444	0.511858	-
{1, 2, 4}	-	0.147176	0.216616	-	0.636208
{1, 3, 4}	-	0.102122	-	0.395562	0.502315
{2, 3, 4}	-	-	0.27787	0.251982	0.470148
{0, 1, 2, 3}	0.064422	0.151858	0.280029	0.503691	-
{0, 1, 2, 4}	0.066709	0.106637	0.215865	-	0.610789
{0, 1, 3, 4}	0.053548	0.08145	-	0.395562	0.469439
{0, 2, 3, 4}	0.046842	-	0.265874	0.251982	0.435302
{1, 2, 3, 4}	-	0.098786	0.192909	0.251982	0.456323
{0, 1, 2, 3, 4}	0.046842	0.078113	0.192158	0.251982	0.430904