Strotz Meets Allais: Diminishing Impatience and the Certainty Effect: Comment

By Kota Saito*

Halevy (2008) states the equivalence between diminishing impatience (i.e., quasi–hyperbolic discounting) and the common ratio effect. The present paper shows that one way of the equivalence is false and shows the correct and general relationships: diminishing impatience is equivalent to the certainty effect and that strong diminishing impatience (i.e., hyperbolic discounting) is equivalent to the common ratio effect.

The main theorem of Yoram Halevy (2008, Theorem 1, p.1150) states that a rank–dependent expected utility maximizer exhibits *diminishing impatience* (i.e., quasi–hyperbolic discounting) if and only if the elasticity of the probability weighting function of the decision maker is increasing. An implication of such a result is the equivalence between diminishing impatience and the *common ratio effect* (CRE).

Claim 1 of this comment shows that the "only if" part of Theorem 1 in Halevy (2008) is false. One might wonder whether, even though the theorem is false, the main implication relating diminishing impatience with the CRE is true. Claim 2 uses the example of Claim 1 to show that diminishing impatience does not imply the CRE.

Given the correction, two natural questions arise when considering a joint model, such as Halevy's (2008), that relates decision under risk and inter–temporal decision: (i) "Is there a behavioral property in decision under risk that is equivalent to diminishing impatience?"; (ii) "Is there a behavioral property of inter–temporal decision making that is equivalent to the CRE?" Claim 3 answers the questions: it shows that diminishing impatience is equivalent to the *certainty effect* (CE); and that *strong diminishing impatience* (i.e., hyperbolic discounting) is equivalent to the CRE.

An implication of Claim 3 is that under any additional assumptions which would make the "only if" part of Theorem 1 in Halevy (2008) true, diminishing impatience becomes equivalent to strong diminishing impatience, and, similarly, the CRE also becomes equivalent to the CE under any such assumption. Thus, whatever assumption is added to make the "only if" part of Theorem 1 true, it must confound the conceptually clear and empirically robust distinction between quasi–hyperbolic discounting and hyperbolic discounting on the one hand, and also between the CRE and the CE on the other hand.

To provide Claim 1, we review the setup used by Halevy (2008). He characterizes

* Department of Economics, Northwestern University, Evanston, IL 60208 (e-mail: kota.saito07@gmail.com). As of July 2011, Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena, CA 91125. I am indebted to my adviser Eddie Dekel for continuous guidance. I am also indebted to my former adviser Michihiro Kandori for his guidance when I was at the University of Tokyo. I am grateful to Jeff Ely and Marciano Siniscalchi for helpful comments. I would also like to thank Kazuki Baba, Colin Camerer, Thomas Epper, Faruk Gul, Yoram Halevy, Daisuke Ikeda, Kazuya Kamiya, David Levine, Akihiko Matsui, Manel Baucells, Shinjiro Miyazawa, Daisuke Nakajima, Arthur Robson, Balazs Szentes, Kosuke Uetake and seminar audiences at Northwestern. I acknowledge financial support from the NSF grant SES 0820333 and the Center for Economic Theory of Northwestern University.

quasi–hyperbolic discounting in terms of *diminishing impatience*:

$$\forall t \in \mathbb{Z}_+, t \geq 1 \ : \ \frac{D(0)}{D(1)} > \frac{D(t)}{D(t+1)}, \tag{1}$$

where $D(\cdot)$ is a discount function. Also, he characterizes hyperbolic discounting in terms of *strong diminishing impatience*:

$$\forall t \in \mathbb{Z}_+ \ : \ \frac{D(t)}{D(t+1)} > \frac{D(t+1)}{D(t+2)}. \tag{2}$$

He studies a decision maker for whom

$$D(t) = \beta^t g((1-r)^t), \tag{3}$$

where β is a pure time–discount factor, g is a rank–dependent probability–weighting function, and r is a constant hazard probability per period.

Theorem 1 in Halevy (2008) is equivalent to the statement that D exhibits diminishing impatience if and only if the elasticity $\varepsilon_g(p) \equiv g'(p)\, p/g(p)$ of g is increasing. Claim 1 shows that the "only if" part of this theorem is false (although it should be noted that this error essentially originates in Uzi Segal (1987a, Lemma 4.1), on which the "only if" part of Theorem 1 in Halevy (2008) crucially depends).

CLAIM 1: *Suppose that*

$$g(p) = \frac{\sqrt{p}}{\left(\sqrt{p} + \sqrt{1-p}\right)^2}. \tag{4}$$

Then $D(t)$ exhibits diminishing impatience while g has decreasing elasticity in a neighborhood of $p = 0$.

PROOF:
First, we will show that D exhibits diminishing impatience. As Halevy (2008, p.1150) shows in Theorem 1, diminishing impatience is equivalent to $g(pq) > g(p)\, g(q)$ for all $p, q \in (0,1)$. That this inequality holds for g defined by (4) follows from the following straightforward calculation. Simple algebra yields that this inequality holds if and only if for all $p, q \in (0,1)$,

$$\sqrt{p(1-p)} + \sqrt{q(1-q)} + 2\sqrt{pq(1-p)(1-q)} > \sqrt{pq(1-pq)}, \tag{5}$$

which is true because $2\sqrt{(1-p)(1-q)} > \sqrt{(1-pq)}$ for all $p, q \in (0,1)$.

Next, we will show g has decreasing elasticity in a neighborhood of $p = 0$. This is also by direct calculation. With some manipulation, the derivative

$$\varepsilon_g'(p) = -\frac{1 - 2p - 2\sqrt{(1-p)p}}{2(\sqrt{1-p} + \sqrt{p})^2 (1-p)^{3/2}\sqrt{p}}, \tag{6}$$

which, one can immediately see, is continuous and converges to $-\infty$ as $p \to 0$. Therefore, $\varepsilon_g'(p)$ is negative in a neighborhood of zero. This completes the proof of Claim 1.

Given Claim 1, Claim 2 shows that diminishing impatience does not imply the CRE. To provide Claim 2, we review the definition of the CRE. Denote by (x, l), a lottery which gives a positive prize $x \in \mathbb{R}_+$ with probability $l \in [0, 1]$ and gives 0 with the rest of probability $1 - l$. The *common ratio effect* (CRE) is defined by Daniel Kahneman and Amos Tversky (1979 p.282) as follows: for all $x, y \in \mathbb{R}_+$ and $p, q \in (0, 1)$, $l \in (0, 1]$,

$$\text{if } (y, ql) \text{ is indifferent to } (x, l), \text{ then } (y, pql) \text{ is preferred to } (x, pl).$$

The *certainty effect* (CE) is defined as a special case of the CRE when $l = 1$.

CLAIM 2: *Rank–dependent risk preferences determined by g defined by (4) together with a continuous and strictly increasing utility function u satisfying $u(0) = 0$ do not exhibit the common ratio effect (CRE).*

PROOF:
This would follow from Claim 1 and Halevy's (2008) claim that Segal (1987b) proves that increasing elasticity holds if and only if the CRE holds. However, Segal (1987b) only proves that increasing elasticity implies the CRE. We show the converse here. Since we showed that the elasticity is decreasing in a neighborhood of 0, there exist p^*, q^* such that $0 < p^* < q^*$ and $\varepsilon_g(p^*) > \varepsilon_g(q^*)$. By defining $\alpha = q^*/p^* > 1$, $g(\alpha p^*)g'(p^*) > \alpha g'(\alpha p^*)g(p^*)$, so that

$$\frac{d(g(\alpha p)/g(p))}{dp}\Big|_{p=p^*} = \frac{\alpha g'(\alpha p^*)g(p^*) - g(\alpha p^*)g'(p^*)}{(g(p^*))^2} < 0. \tag{7}$$

This is true for any p in the region where the elasticity is decreasing. Hence, by the continuity of g, there exist p' and q' in the region where the elasticity is decreasing such that $p' < q'$, $\alpha q' < 1$, and $g(\alpha p')/g(p') > g(\alpha q')/g(q')$. Since u is strictly increasing and $u(0) = 0$, $\lim_{x\to 0} u(x)/u(y) = 0 < g(\alpha p')/g(\alpha q') < 1 = u(y)/u(y)$ for all $y > 0$. Therefore, by the continuity of u, there exist prizes x and y such that

$$\frac{g(\alpha p')}{g(\alpha q')} = \frac{u(x)}{u(y)} > \frac{g(p')}{g(q')}. \tag{8}$$

This implies that $g(\alpha q')u(x) = g(\alpha p')u(y)$ but $g(q')u(x) > g(p')u(y)$. Since $\alpha > 1$, this violates the CRE and completes the proof of Claim 2.

Claims 1 and 2 imply that only a *partial* result of Halevy (2008) is true in general: The CRE implies diminishing impatience. Claim 3 shows the *complete* relationships as follows:

CLAIM 3: *Suppose that D is defined by (3).*
(i) Diminishing impatience is equivalent to the certainty effect (CE),
(ii) Strong diminishing impatience is equivalent to the common ratio effect (CRE).

PROOF:
For the rank–dependent utility maximizer, the CE is as follows: for all outcomes x, y and $p, q \in (0, 1)$, if $u(x) = g(q)u(y)$ then $g(p)u(x) < g(pq)u(y)$. Equivalently, for all

$p, q \in (0,1)$, $g(p)g(q) < g(pq)$. Therefore,

$$
\begin{array}{l}
\text{Diminishing Impatience} \\
\Leftrightarrow \forall t \in \mathbb{Z}_+, \forall r \in (0,1)\ \left[g((1-r)^{t+1}) > g(1-r)g((1-r)^t)\right] \\
\Leftrightarrow \forall p, q \in (0,1)\ \left[g(pq) > g(p)g(q)\right] \\
\Leftrightarrow \text{Certainty Effect,}
\end{array} \tag{9}
$$

where the first and the second equivalences are by Theorem 1 of Halevy (2008, p.1150).

Similarly, the CRE is as follows: for all outcomes x, y and $p, q \in (0,1)$, $l \in (0,1]$, if $g(l)u(x) = g(ql)u(y)$ then $g(pl)u(x) < g(pql)u(y)$. Equivalently, for all $p, q \in (0,1)$, $l \in (0,1]$, $g(pl)g(ql) < g(l)g(pql)$. Therefore,

$$
\begin{array}{l}
\text{Strong Diminishing Impatience} \\
\Leftrightarrow \forall t \in \mathbb{Z}_+,\ \forall s \in \mathbb{Z}_{++},\ \forall r \in (0,1)\ \left[\frac{D(t)}{D(t+1)} > \frac{D(t+s)}{D(t+s+1)}\right] \\
\Leftrightarrow \forall t \in \mathbb{Z}_+,\ \forall s \in \mathbb{Z}_{++},\ \forall r \in (0,1)\ \left[g((1-r)^t)g((1-r)^{t+s+1}) > g((1-r)^{t+1})g((1-r)^{t+s})\right] \\
\Leftrightarrow \forall p, q \in (0,1), \forall l \in (0,1]\ \left[g(l)g(pql) > g(pl)g(ql)\right] \\
\Leftrightarrow \text{Common Ratio Effect.}
\end{array} \tag{10}
$$

This completes the proof of Claim 3.

There are assumptions, such as monotonicity of $\varepsilon_g(p)$ or convexity of g, under which the "only if" part of Theorem 1 in Halevy (2008) is true, so that diminishing impatience becomes equivalent to the CRE. As noted earlier, Claim 3 shows that any such assumption must confound the distinction between quasi–hyperbolic discounting and hyperbolic discounting, and also between the CRE and the CE.

Claim 3 is a special case of the results in Kota Saito (2009) which shows these relationships without assuming specific form of utility function and hazard probability function.

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