Online Appendix: Preferences for Flexibility and Randomization under Uncertainty

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In this online appendix, we present an extension of Theorem 1. We show that an RUA representation can incorporate the variational preferences of Maccheroni, Marinacci, and Rustichini (2006), which are general enough to incorporate other well-known uncertainty-averse preferences, such as the maxmin preferences of Gilboa and Schmeidler (1989), the multiplier preferences of Hansen and Sargent (2001), and the vector-expected utility preferences of Siniscalchi (2009).[1] We present the extension of Theorem 1 in Section 1. The proof is in Section 2. In Section 3, we provide the omitted proofs of Lemmas 1 and 2 in the proof of Theorem 1.

1 Extension

We assume that $\mathscr{U} = \mathbb{R}$. This assumption is equivalent to the unboundedness axiom proposed by Maccheroni et al. (2006). Instead of the certainty set independence axiom, we use the following weaker axiom:

Axiom (Weak Certainty Set Independence):

$$\alpha A+(1-\alpha)(x,\dots,x) \succsim \alpha B+(1-\alpha)(x,\dots,x) \Leftrightarrow \alpha A+(1-\alpha)(y,\dots,y) \succsim \alpha B+(1-\alpha)(y,\dots,y).$$

[1] The vector-expected utility preferences that satisfy the quasi-concavity axiom are a special case of the variational preferences.

This axiom is a natural extension of Maccheroni et al.'s (2006) weak certainty independence axiom. This axiom means that the agent can eliminate the effects of uncertainty by mixing an act with a constant act.[2] Consequently, the agent could violate the certainty strategic rationality axiom. For example, even if the agent has the option to choose $(x, ..., x)$ and he prefers $(x, ..., x)$ to $\{g, h\}$, he may prefer to have the additional option to choose g or h because he may eliminate the effects of uncertainty by randomizing $(x, ..., x)$ with either g and h.

We will propose an axiom, *weak strategic rationality*, that allows the above preference. The axiom can be stated as follows: if the agent has the option to choose f, he prefers f to $\{g, h\}$, and he does not have a strict incentive to randomize f with either g and h, then he is indifferent to having the additional option to choose g or h.

Definition 7: For all $f, g, h \in \mathscr{F}$, we say that *the agent does not have a strict incentive to randomize f with g and h*, if for all $\mu \in \Delta(\{f, g, h\})$ there exists $\rho \in \Delta(\{f, g, h\})$ such that (i) ρ dominates μ and

$$\text{(ii)} \qquad \rho_1 f + \rho_2 g + \rho_3 h \sim \rho_1 x(f) + (1-\rho_1)x\Big(\frac{\rho_2}{1-\rho_1}g + \frac{\rho_3}{1-\rho_1}h\Big),$$

where $\rho = \rho_1 f \oplus \rho_2 g \oplus \rho_3 h$.

The condition (i) means that the agent could choose ρ over μ regardless of whether he believes that his randomizations eliminate the effects of uncertainty. To interpret (ii), consider the case of two acts (i.e., $g = h$); then (ii) reduces to $\rho_1 f + (1-\rho_1)g \sim \rho_1 x(f) + (1-\rho_1)x(g)$. Recall that the strict uncertainty aversion implies that $\rho_1 f + (1-\rho_1)g \succ \rho_1 x(f) + (1-\rho_1)x(g)$. Hence, the indifference (i.e., $\rho_1 f + (1-\rho_1)g \sim \rho_1 x(f) + (1-\rho_1)x(g)$) means that mixing f and g does not eliminate the effects of uncertainty. Condition (ii) is merely an extension of the case of three acts. Note that, however, in (ii), we evaluate the mixture of g and h jointly (i.e., $(1-\rho_1)x\big(\frac{\rho_2}{1-\rho_1}g + \frac{\rho_3}{1-\rho_1}h\big)$) rather than separately (i.e., $\rho_2 x(g) + \rho_3 x(h)$). This is because what matters here is the *additional* contribution by mixing f with g and h.

[2] Maccheroni et al.'s (2006) weak certainty independence axiom is as follows: $\alpha f + (1-\alpha)(x, \ldots, x) \succsim \alpha g + (1-\alpha)(x, \ldots, x) \Leftrightarrow \alpha f + (1-\alpha)(y, \ldots, y) \succsim \alpha g + (1-\alpha)(y, \ldots, y)$.

In summary, (i) and (ii) mean that for any randomization μ over $\{f, g, h\}$, there exists a dominant randomization ρ in which mixing f with g and h does not eliminate the effects of uncertainty. Hence, the agent does not have a strict incentive to randomize f with g and h. Consequently, if the agent has the option to choose f and he prefers f to $\{g, h\}$, then he should be indifferent to having the additional option to choose g or h. This suggests the following axiom:[3]

Axiom (Weak Strategic Rationality): Suppose that the agent does not have a strict incentive to randomize f with g and h. Then,

$$\{f\} \succsim \{g, h\} \Rightarrow \{f\} \sim \{f, g, h\}.$$

We need to introduce one more axiom: *indifference*. To explain this axiom, remember the two urns in Section 2 and suppose that the agent can determine the ball color drawn from the uncertain urn on which he bets. Then, the agent's choice set is $\{f^{\text{Red}}, f^{\text{Black}}\}$. As we discussed in Section 4, when the agent believes that his randomization eliminates the effects of uncertainty, he should be indifferent between $\{f^{\text{Red}}, f^{\text{Black}}\}$ and its convex hull $\text{co}(\{f^{\text{Red}}, f^{\text{Black}}\})$. On the other hand, when the agent believes that his randomization does not eliminate the effects of uncertainty, he should be indifferent between $\{f^{\text{Red}}, f^{\text{Black}}\}$ and the singleton of each act (i.e., $\{f^{\text{Red}}\}$ or $\{f^{\text{Black}}\}$).

The indifference axiom can be explained as follows: Consider two sets $\{f, g\}$ and $\{f', g'\}$ in regard to which the agent is indifferent between the two acts in either set (i.e., $f \sim g$ and $f' \sim g'$). Suppose that the agent's utility of the set $\{f, g\}$ is half the utility of the set $\{f', g'\}$ (i) when he believes that his randomizations eliminate the effects of uncertainty, and (ii) when he believes that his randomizations do not eliminate that effects. Then, the indifference axiom requires that his utility of the set $\{f, g\}$ is half the utility of the set $\{f', g'\}$. To formalize the indifference axiom, we need a new notation: $x(A) \in \mathscr{U}$ is the certainty equivalent of set A

[3]Under the certainty set independence axiom, the weak strategic rationality axiom implies the certainty strategic rationality axiom.

(i.e., $x(A) \sim A$).

Axiom (Indifference): Suppose that $f \sim g$ and $f' \sim g'$.

(i) $x\big(\mathrm{co}(\{f,g\})\big) = \frac{1}{2}x\big(\mathrm{co}(\{f',g'\})\big)$, and (ii) $x(f) = \frac{1}{2}x(f') \Rightarrow x\big(\{f,g\}\big) = \frac{1}{2}x\big(\{f',g'\}\big)$.

Note that conditions (i) and (ii) mean that the utility of the set $\{f, g\}$ is half the utility of the set $\{f', g'\}$ (i) when he believes that his randomizations eliminate the effects of uncertainty, and (ii) when he believes that his randomizations do not eliminate that effects, respectively.

Theorem 2: *$\succsim$ satisfies Weak Order, Continuity, Weak Certainty Set Independence, Uncertainty Aversion, Weak Strategic Rationality, Dominance, and Indifference if and only if there exists a pair (δ, c) of $\delta \in [0,1]$ and $c : \Delta(S) \to \mathbb{R}_+$ such that $\succsim$ is represented by a function $U : \mathscr{A} \to \mathbb{R}$ defined by*

$$U(A) = \max_{\rho \in \Delta(A)} \Big[\delta \; u\big(\sum_{f \in \mathscr{F}} \rho(f) f\big) + (1-\delta) \sum_{f \in \mathscr{F}} \rho(f) u\big(f\big)\Big], \tag{1}$$

where $u(f) = \min_{p \in \Delta(S)} \Big(\sum_{s \in S} p(s) f(s) + c(p)\Big)$ and c is a grounded, convex, and lower semicontinuous function. The function c is unique. Moreover, δ is unique if $c(p) < \infty$ and $c(q) < \infty$ for some $p, q \in \Delta(S)$ such that $p \neq q$.

The function c is said to be *grounded* if its infimum value is zero. It can be shown that δ is unique as long as the agent is not an expected utility maximizer.[4] Note that the representations in Theorem 1 and 2 differ only in u. The other mathematical structures are the same. Hence, even with this extended representation, we can perform essentially the same comparative statics and application.

[4] It can be shown that there exist $p, q \in \Delta(S)$ such that $p \neq q$, $c(p) < \infty$, and $c(q) < \infty$, if and only if there exist $f, g \in \mathscr{F}$ and $\alpha \in [0,1]$ such that $f \sim g$ and $\alpha f + (1-\alpha) g \succ f$.

2 Proof of Theorem 2

In the following, we present the proof of Theorem 2. Remember the following notations. We denote a singleton $\{f\}$ by f. We denote constant acts $(x, ..., x)$ and $(y, ..., y)$ by x and y when there is no danger of confusion. For example, we denote $\alpha f + (1-\alpha)(x, \ldots, x)$ by $\alpha f + (1-\alpha)x$.

2.1 Proof of Sufficiency

The proof for sufficiency consists of nine lemmas. First, we present the outline of the proof with the statements of the lemmas. After that, we present the proofs of the lemmas. We omit the proofs of several lemmas which are the same as in the proof of Theorem 1.

Fix $\succsim$ on $\mathscr{A}$ that satisfies the all axioms in Theorem 2. By using results in Maccheroni et al. (2006), we obtain the first lemma.

Lemma 1 *There exist a function $U : \mathscr{A} \to \mathbb{R}$ and a grounded, convex, and lower semi-continuous function $c : \Delta \to [0, +\infty]$ such that (i) $U(A) \geq U(B) \Leftrightarrow A \succsim B$; (ii) $U(f) = \min_{p \in \Delta(S)} \sum_{s \in S} p(s) f(s) + c(p)$ for all $f \in \mathscr{F}$. Moreover, c unique and U is concave and continuous on $\mathscr{F}$.*

Lemmas 2 and 3 are the same as in the proof of Theorem 1 and the proofs are omitted.

Lemma 2 *(i) For all $\rho \in \Delta(\mathscr{F})$, $U(\sum_{f \in \mathscr{F}} \rho(f) f) \geq \sum_{f \in \mathscr{F}} \rho(f) U(f)$; and*
(ii) $\max_{\rho \in \Delta(A)} U(\sum_{f \in \mathscr{F}} \rho(f) f) \geq U(A) \geq \max_{\rho \in \Delta(A)} \sum_{f \in \mathscr{F}} \rho(f) U(f)$.

Lemma 3 *Suppose that U has an expected utility representation on $\mathscr{F}$. Then, for any $\delta \in [0, 1]$ and $A \in \mathscr{A}$, $U(A) = \max_{\rho \in \Delta(A)} \delta U(\sum_{f \in \mathscr{F}} \rho(f) f) + (1-\delta) \sum_{f \in \mathscr{F}} \rho(f) U(f)$.*

Lemma 3 establishes Theorem 2 in the case where U has an expected utility representation on $\mathscr{F}$. In the following, we consider the case where U does not have an expected utility representation on $\mathscr{F}$. Define $\mathscr{L}$ and $\delta : \mathscr{L} \to \mathbb{R}$ as in the proof of Theorem 1. The outline of the proof is the same as in the proof of Theorem 1. First, we obtain the desired representation on $\mathscr{L}$. Then, we extend the representation on $\mathscr{A}$.

We show four preliminary results. We define notations used in the following: for all $f \in \mathscr{F}$ and $x \in \mathscr{U}$, $f(s) + x \in \mathscr{U}$ for each $s \in S$. Hence, $(f(1) + x, \ldots, f(n) + x) \in \mathscr{F}$. We define $f + x = (f(1) + x, \ldots, f(n) + x)$. For all $A \in \mathscr{A}$ and $x \in \mathscr{U}$, define $A + x = \{f + x | f \in A\}$. The notations are well defined because $\mathscr{U} = \mathbb{R}$. Lemma 4 follows from the property of the variational preferences and the definition of δ.

Lemma 4 *(i) For all $f \in \mathscr{F}$ and $x \in \mathscr{U}$, $U(f + x) = U(f) + x$; (ii) For all $A \in \mathscr{A}$ and $x \in \mathscr{U}$, $U(A+x) = U(A)+x$; (iii) for all $\{f, g\} \in \mathscr{L}$ and $x \in \mathbb{R}$, $\delta(\{f, g\}) = \delta(\{f+x, g+x\})$.*

Lemma 5 follows from the property of the variational preferences and the assumption that $\mathscr{U} = \mathbb{R}$.

Lemma 5 *For any $c \in \mathbb{R}$ and $d \in \mathbb{R}_+$, there exist $f, g \in \mathscr{F}$ such that $f \sim g$, $U(f) = c$, and $\max_{\alpha \in [0,1]} U(\alpha f + (1 - \alpha)g) - U(f) = d$.*

Lemma 6 follows from the property of the variational preferences.

Lemma 6 *Suppose that $f, g \in \mathscr{F}$, $f \sim g$, and $\alpha^* \in \arg\max_{\alpha \in [0,1]} U(\alpha f + (1 - \alpha)g)$. For any $y \in \mathbb{R}$, the agent does not have a strict incentive to randomize $(\alpha^* f + (1 - \alpha^*)g) + y$ with f and g.*

Lemma 7 is similar to Lemma 8 in the proof of Theorem 1, so the proof is omitted.

Lemma 7 *Suppose that $f, g \in \mathscr{F}$, $f \sim g$, $U(f) = c$, and $\max_{\alpha \in [0,1]} U(\alpha f+(1-\alpha)g)-U(f) = d$.*
(i) $\{f, g\}^* = co(\{(c, c), (c + d, c)\})$,
(ii) If $\alpha^* \in \max_{\alpha \in [0,1]} U(\alpha f + (1 - \alpha)g)$ and $y \in \mathscr{U}$, then $\{f, g, (\alpha^* f + (1 - \alpha^*)g) + y\}^* = co(\{(c, c), (c + d, c), (c + d + y, c + d + y)\})$.

Define v_1, v_2, $\boldsymbol{v}$, and A^* for all $A \in \mathscr{A}$ as in the proof Theorem 1. The next lemma is the same as in the proof of Theorem 1, so the proof is omitted.

Lemma 8 *(i) If A^* dominates B^*, then $U(A) \geq U(B)$; (ii) If $A^* \supset B^*$ then $U(A) \geq U(B)$.*

By using Lemma 5 and 6, we obtain the desired representation on $\mathscr{L}$ in the next lemma.

Lemma 9 *There exists $\delta \in [0,1]$ such that $U(\{f,g\}) = \max_{\rho \in \Delta(\{f,g\})} \delta U(\sum_{f \in \mathscr{F}} \rho(f)f) + (1-\delta)\sum_{f \in \mathscr{F}} \rho(f)U(f)$ for all $\{f,g\} \in \mathscr{L}$.*

By using the above lemmas, we obtain the desired representation on $\mathscr{A}$.

Lemma 10 *For all $A \in \mathscr{A}$, $U(A) = \max_{\rho \in \Delta(A)} \delta U(\sum_{f \in \mathscr{F}} \rho(f)f) + (1-\delta)\sum_{f \in \mathscr{F}} \rho(f)U(f)$.*

In the following, we present the proofs of the above lemmas.

Proof of Lemma 1: Note that Weak Certainty Set Independence implies the the weak certainty independence axiom. By identifying $\{f\}$ as f for all $f \in \mathscr{F}$, we can confirm that $\succsim$ satisfies all the conditions of Theorem 3 of Maccheroni et al. (2006, p.1456). Hence, there exists a grounded, convex, and lower semicontinuous function $c : \Delta \to [0, +\infty]$ such that $u(f) \equiv \min_{p \in \Delta(S)} \sum_{s \in S} p(s)f(s) + c(p)$ represents $\succsim$ on $\mathscr{F}$. Moreover, the assumption that $\mathscr{U} = \mathbb{R}$ is equivalent to the unboundedness axiom in Maccheroni et al. (2006). It follows from Proposition 6 of Maccheroni et al. (2006, p.1457) that c is unique.

By Dominance and Continuity, for any $A \in \mathscr{A}$, we can find $f \in \mathscr{F}$ such that $\{f\} \sim A$. Define $U(A) = u(f)$. So we obtain (i) and (ii).

A direct calculation shows that u is concave.[5] Since $\mathscr{F} \equiv \mathbb{R}^S$ is open, therefore, u is continuous. Hence, U is concave and continuous on $\mathscr{F}$.

Proof of Lemma 4: To show (i), fix $f \in \mathscr{F}$ and $x \in \mathscr{U}$, $U(f+x) = \min_{p \in \Delta(S)} \sum_{s \in S} p(s)(f(s)+x) + c(p) = (\min_{p \in \Delta(S)} \sum_{s \in S} p(s)f(s) + c(p)) + x = U(f) + x$.

To show (ii), fix $A \in \mathscr{A}$ and $x \in \mathscr{U}$. By Dominance and Continuity, there exists $f \in \mathscr{F}$ such that $f \sim A$. Hence, by (i), $U(A+x) = U(f+x) = U(f) + x = U(A) + x$.

[5]Fix $f, g \in \mathscr{F}$ and $\alpha \in [0,1]$. Let $p^* \in \arg\min_{p \in \Delta(S)} \sum_{s \in S} p(s)(\alpha f(s) + (1-\alpha)g(s)) + c(p)$. (Such minimizer exists because c is lower semicontinuous and $\Delta(S)$ is compact.) Then, $u(\alpha f + (1-\alpha)g) = \alpha(\sum_{s \in S} p^*(s)f(s) + c(p^*)) + (1-\alpha)(\sum_{s \in S} p^*(s)g(s) + c(p^*)) \geq \alpha u(f) + (1-\alpha)u(g)$.

Finally to show (iii), let $\alpha^* \in \arg\max_{\alpha\in[0,1]} U(\alpha f+(1-\alpha)g)$. Since $U(\alpha(f+x)+(1-\alpha)(g+x)) = U(\alpha f+(1-\alpha)g)+x$ for all $\alpha \in [0,1]$, then $\alpha^* \in \arg\max_{\alpha\in[0,1]} U(\alpha(f+x)+(1-\alpha)(g+x))$. Moreover, by (ii), $U(\{f+x, g+x\}) = U(\{f,g\}) + x$. So,

$$\begin{aligned}\delta(\{f+x, g+x\}) &= \frac{U(\{f+x,g+x\}) - U(f+x)}{U(\alpha^*(f+x)+(1-\alpha^*)(g+x)) - U(f+x)} \\ &= \frac{U(\{f,g\}) - U(f)}{U(\alpha^* f + (1-\alpha^*)g) - U(f)} \\ &= \delta(\{f,g\}).\end{aligned}$$

Proof of Lemma 5: Since U does not have an expected utility representation, there exist $p, q \in \Delta(S)$ such that $p \neq q$, $c(p) < \infty$, and $c(q) < \infty$. Since $p \neq q$, there exist $s_1, s_2 \in S$ such that $p(s_1) > p(s_2)$ and $q(s_1) < q(s_2)$. It can be shown that there exist $x > 0$, $f', g' \in \mathscr{F}$ such that $(x,\dots,x) \equiv \frac{1}{2}f' + \frac{1}{2}g' \succ f' \sim g'$.[6]

Fix $c \in \mathbb{R}$ and $d \in \mathbb{R}_+$. For all $a \in \mathbb{R}_+$, define $af' = (af'(s))_{s\in S}$ and $ag' = (ag'(s))_{s\in S}$. For all $a \in \mathbb{R}_+$, define $d(a) = \max_{\alpha\in[0,1]} U\big(\alpha af' + (1-\alpha)ag'\big) - \big(\alpha U(af') + (1-\alpha)U(ag')\big)$. Since $[0,1]$ is compact and u is continuous, Berge's maximum theorem shows that d is continuous. Obviously, $d(a) \to 0$ as $a \to 0$. By a direct calculation for any $a > 1$, $aU(f') \geq U(af')$ and $aU(g') \geq U(ag')$.[7] Hence, $d(a) \geq U\big(\frac{1}{2}af' + \frac{1}{2}ag'\big) - (\frac{1}{2}U(af') + \frac{1}{2}U(ag')) \geq U\big(\frac{1}{2}af' + \frac{1}{2}ag'\big) - a(\frac{1}{2}U(f')+\frac{1}{2}U(g')) = a(x-U(f'))$, where the equality holds because $\frac{1}{2}af'+\frac{1}{2}ag' = (ax,\dots,ax)$. Since $x - U(f') > 0$, $d(a) \to +\infty$ as $a \to +\infty$. Hence, by the intermediate value theorem, there exists $a^* \in \mathbb{R}_+$ such that $d(a^*) = d$.

Define $x = c - U(a^*f')$ and $y = c - U(a^*g')$. Define $f = a^*f' + x$ and $g = a^*g' + y$. Then, $U(f) = c = U(g)$. Moreover, for all $\alpha \in [0,1]$, $\alpha U(f) + (1-\alpha)U(g) = (\alpha U(a^*f') + (1-\alpha)U(a^*g')) + (\alpha x + (1-\alpha)y)$ and $U\big(\alpha f + (1-\alpha)g\big) = U\big(\alpha a^* f' + (1-\alpha)a^*g'\big) + (\alpha x + (1-\alpha)y)$. Hence, $U\big(\alpha f + (1-\alpha)g\big) - \big(\alpha U(f) + (1-\alpha)U(g)\big) = U\big(\alpha a^* f' + (1-\alpha)a^*g'\big) - \big(\alpha U(a^*f') + (1-$

[6] There exist positive numbers ε, σ such that $0 > -p(s_1)\sigma + p(s_2)\varepsilon + c(p) = q(s_1)\varepsilon - q(s_2)\sigma + c(q)$. This shows $\frac{1}{2}(-\sigma,\varepsilon,\mathbf{0}) + \frac{1}{2}(\sigma,-\varepsilon,\mathbf{0}) = (0,\dots,0) \succ (-\sigma,\varepsilon,\mathbf{0}) \sim (\sigma,-\varepsilon,\mathbf{0})$, where $(x,y,\mathbf{0})$ denotes an act that yields x at state s_1; yields y at state s_2; and yields 0 for the other states. Adding constant $(x,\dots,x)$ to each act yields the desired result.

[7] Let $p^* \in \arg\min_{p\in\Delta(S)} \sum p(s)f(s) + c(p)$. Then, $aU(f) = au(f) = a(\sum p^*(s)f(s) + c(p^*)) \geq \sum p^*(s)af(s) + c(p^*) \geq \min_{p\in\Delta(S)} \sum p(s)af(s) + c(p) = U(af)$, as desired.

$\alpha)U(a^*g')\big)$ for all $\alpha \in [0,1]$. Therefore, $\max_{\alpha\in[0,1]} U\big(\alpha f+(1-\alpha)g\big)-\big(\alpha U(f)+(1-\alpha)U(g)\big) = d(a^*) = d$.

Proof of Lemma 6: Define $h^* = \alpha^* f + (1-\alpha^*)g$ and fix any $y \in \mathbb{R}$. To show that the agent does not have a strict incentive to randomize $h^* + y$ with f and g, choose any $\mu \in \Delta(\{h^*+y, f, g\})$ such that $\mu = \mu_1(h^*+y) \oplus \mu_2 f \oplus \mu_3 g$. First, we show that $\mu_1(h^* + y) \oplus (1-\mu_1)\alpha^* f \oplus (1-\mu_1)(1-\alpha^*)g$ dominates μ. Since $x(f) = x(g)$, then $\mu_1 x(h^* + y) + \mu_2 x(f) + \mu_3 x(g) = \mu_1 x(h^*+y) + (1-\mu_1)\alpha^* x(f) + (1-\mu_1)(1-\alpha^*)x(g)$. In addition, $x\big(\mu_1(h^*+y)+\mu_2 f+\mu_3 g\big) = x\big((\mu_1\alpha^*+\mu_2)f+(\mu_1(1-\alpha^*)+\mu_3)g\big)+\mu_1 y \le x(h^*)+\mu_1 y = x\big(\mu_1(h^*+y)+(1-\mu_1)\alpha^* f+(1-\mu_1)(1-\alpha^*)g\big)$, where the first equality holds because $h^* = \alpha^* f + (1-\alpha^*)g$ and the inequality holds because $\alpha^* \in \arg\max_{\alpha\in[0,1]} x(\alpha f + (1-\alpha)g)$. Hence, $\mu_1(h^*+y) \oplus (1-\mu_1)\alpha^* f \oplus (1-\mu_1)(1-\alpha^*)g$ dominates μ.

Finally, note that $\mu_1 x\big(h^*+y\big)+(1-\mu_1)x\big(\alpha^* f+(1-\alpha^*)g\big) = x\big(\alpha^* f+(1-\alpha^*)g\big)+\mu_1 y = x\big(\mu_1(h^*+y)+(1-\mu_1)\alpha^* f+(1-\mu_1)(1-\alpha^*)g\big)$. So, the agent does not have a strict incentive to randomize h^*+y with f and g.

Proof of Lemma 9: By the definition of δ, it suffices to show that δ is constant.

Step 1: For any $\{f,g\},\{f',g'\} \in \mathscr{L}$, if $\max_{\alpha\in[0,1]} U(\alpha f+(1-\alpha)g)-U(f) = \max_{\alpha\in[0,1]} U(\alpha f'+(1-\alpha)g')-U(f')$ then $\delta(\{f,g\}) = \delta(\{f',g'\})$.

Proof of Step 1: Define $x = U(f) - U(f')$. Then, $x = U(g) - U(g')$. By Lemma 4 (ii), $U(\{f'+x, g'+x\}) - U(f'+x) = U(\{f,g\}) - U(f)$. Moreover, by Lemma 4 (i) and the assumption, $\max_{\alpha\in[0,1]} U(\alpha f+(1-\alpha)g)-U(f) = \max_{\alpha\in[0,1]} U(\alpha(f'+x)+(1-\alpha)(g'+x)) - U(f'+x)$. Hence,

$$\begin{aligned}
\delta(\{f'+x, g'+x\}) &= \frac{U(\{f'+x,g'+x\})-U(f'+x)}{\max_{\alpha\in[0,1]} U(\alpha(f'+x)+(1-\alpha)(g'+x))-U(f'+x)} \\
&= \frac{U(\{f,g\})-U(f)}{\max_{\alpha\in[0,1]} U(\alpha f+(1-\alpha)g)-U(f)} \\
&= \delta(\{f,g\}).
\end{aligned}$$

By Lemma 4 (iii), $\delta(\{f'+x, g'+x\}) = \delta(\{f', g'\})$. So, we obtain $\delta(\{f', g'\}) = \delta(\{f, g\})$.

For any $d > 0$, define $\hat{\delta}(d) = \delta(\{f, g\})$, where $d = \max_{\alpha \in [0,1]} U(\alpha f + (1-\alpha)g) - U(f)$ and $\{f, g\} \in \mathscr{L}$. $\hat{\delta}$ is well defined because of Step 1 and Lemma 5.

Step 2: $\hat{\delta}$ is weakly increasing.

Proof of Step 2: Suppose that there exist $c, d \in \mathbb{R}_+$ such that $d > c > 0$ and $\hat{\delta}(d) < \hat{\delta}(c)$. By Lemma 5, there exist $f, g, f', g' \in \mathscr{F}$ such that $U(f) = 0 = U(g)$, $\max_{\alpha \in [0,1]} U(\alpha f + (1-\alpha)g) - U(f) = d$, $U(f') = \hat{\delta}(d)(d-c) = U(g')$, and $\max_{\alpha \in [0,1]} U(\alpha f' + (1-\alpha)g') - U(f') = c$. By the definition of $\hat{\delta}$, $U(\{f, g\}) = \hat{\delta}(d)d$ and $U(\{f', g'\}) = \hat{\delta}(c)(c + \hat{\delta}(d)(d-c)) + (1-\hat{\delta}(c))\hat{\delta}(d)(d-c) = \hat{\delta}(d)(d-c) + \hat{\delta}(c)c = \hat{\delta}(d)d + (\hat{\delta}(c) - \hat{\delta}(d))c$.

Let $\alpha^* \in \arg\max_{\alpha \in [0,1]} U(\alpha f + (1-\alpha)g)$. Define $h^* = \alpha^* f + (1-\alpha^*)g$ and $x = U(\{f, g\}) - U(h^*)$. Then, $h^* + x \sim \{f, g\}$ because $U(h^* + x) = U(\{f, g\})$. By Lemma 6, the agent does not have a strict incentive to randomize $h^* + x$ with f and g. It follows from Weak Strategic Rationality that $h^* + x \sim \{h^* + x, f, g\}$. Then, $U(\{h^* + x, f, g\}) = U(h^* + x) = U(h^*) + x = U(\{f, g\}) = \hat{\delta}(d)d$.

Define $C = \{f', g'\}$ and $D = \{h^* + x, f, g\}$. Then, $U(C) = \hat{\delta}(d)d + (\hat{\delta}(c) - \hat{\delta}(d))c$ and $U(D) = \hat{\delta}(d)d$. By Lemma 7, $C^* = \text{co}(\{(\hat{\delta}(d)(d-c), \hat{\delta}(d)(d-c)), (\hat{\delta}(d)(d-c) + c, \hat{\delta}(d)(d-c))\}) \subset \text{co}(\{(0,0), (d,0), (\hat{\delta}(d)d, \hat{\delta}(d)d)\}) = D^*$ as shown in Figure 1. So, by Lemma 8 (ii), $\hat{\delta}(d)d + (\hat{\delta}(c) - \hat{\delta}(d))c = U(C) \leq U(D) = \hat{\delta}(d)d$, which is a contradiction because $\hat{\delta}(c) > \hat{\delta}(d)$ and $c > 0$.

Step 3: For any $d > 0$ and any positive integer n, $\hat{\delta}(d) = \hat{\delta}(d/2^n)$.

Proof of Step 3: Fix $d > 0$. We prove this step by the induction on n by using the axiom of Indifference. Let $n = 1$. By Lemma 5, there exist $f, g, f', g' \in \mathscr{F}$ such that $f \sim g$, $f' \sim g'$, $U(f) = 0 = U(f')$, $\max_{\alpha \in [0,1]} U(\alpha f + (1-\alpha)g) - U(f) = d$, and $\max_{\alpha \in [0,1]} U(\alpha f' + (1-\alpha)g') - U(f') = d/2$. Lemma 2 (ii) implies that $U(\text{co}(\{f, g\})) = \max_{\alpha \in [0,1]} U(\alpha f + (1-\alpha)g)$ and $U(\text{co}(\{f', g'\})) = \max_{\alpha \in [0,1]} U(\alpha f' + (1-\alpha)g')$. It follows that $\frac{1}{2}U(\text{co}(\{f, g\})) = U(\text{co}(\{f', g'\}))$. Since $\frac{1}{2}U(f) = 0 = U(f')$, Indifference shows that $\frac{1}{2}U(\{f, g\}) = U(\{f', g'\})$.

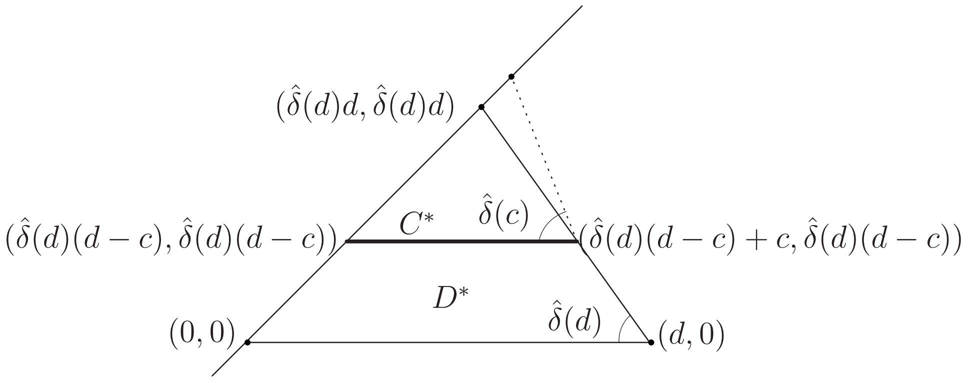

Figure 1: Sets C^* and D^* in Step 2 of Lemma 9

Therefore,

$$\begin{aligned}
\hat{\delta}(d) &= \delta(\{f,g\}) \\
&= \frac{U(\{f,g\}) - U(f)}{\max_{\alpha\in[0,1]} U(\alpha f + (1-\alpha)g) - U(f)} \\
&= \frac{2U(\{f',g'\}) - 2U(f')}{2\max_{\alpha\in[0,1]} U(\alpha f' + (1-\alpha)g') - 2U(f')} \\
&= \delta(\{f',g'\}) \\
&= \hat{\delta}(d/2).
\end{aligned}$$

Now choose any positive integer m. Suppose that $\hat{\delta}(d) = \hat{\delta}(d/2^m)$. By replacing d with $d' = d/2^m$ in the above argument, we can show $\hat{\delta}(d') = \hat{\delta}(d'/2)$. Hence, we obtain $\hat{\delta}(d) = \hat{\delta}(d/2^{m+1})$. This completes the proof of Step 3.

Finally, to see that $\hat{\delta}$ is constant, suppose not. Since $\hat{\delta}$ is weakly increasing, there exist $c, d \in \mathbb{R}_+$ such that $d > c > 0$ such that $\hat{\delta}(d) > \hat{\delta}(c)$. Then, there exist positive integers m, n such that $d/2^m < c/2^n$. By Step 3, $\hat{\delta}(d/2^m) = \hat{\delta}(d) > \hat{\delta}(c) = \hat{\delta}(c/2^n)$. However, this contradicts that $\hat{\delta}$ is weakly increasing.

Proof of Lemma 10: Choose any $A \in \mathscr{A}$.

Case 1: First, we consider the case in which $\delta = 0$. Let $\boldsymbol{v}^* = \arg\max_{\boldsymbol{v}\in A^*} v_2$. (Such $\boldsymbol{v}^*$ exists because v_2 is continuous and A^* is compact.) Let $d^* = \arg\max_{\boldsymbol{v}\in A^*} v_1 - v_2$. By Lemma 5, there exist $f, g \in \mathscr{F}$ such that $U(f) = v_2^* = U(g)$ and $\max_{\alpha\in[0,1]} U(\alpha f + (1-\alpha)g) - U(f) = d^*$. Let $B = \{f, g\}$. By Lemma 9 and $\delta = 0$, $U(B) = v_2^*$. By Lemma 2 (ii), $U(B) = v_2^* \equiv \max_{\boldsymbol{v}\in A^*} v_2 \leq U(A)$.

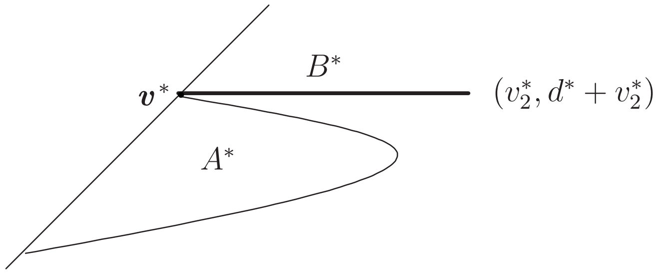

Figure 2: Sets A^* and B^* in Case 1

Next, we show that $U(B) \geq U(A)$. For all $\boldsymbol{v} \in A^*$, $v_1 \leq d^* + v_2^*$ and $v_2 \leq v_2^*$ because $d^* = \max_{\boldsymbol{v}\in A^*} v_1 - v_2$ and $v_2^* = \max_{\boldsymbol{v}\in A^*} v_2$. Moreover, by Lemma 7 (i), $B^* = \mathrm{co}(\{\,\boldsymbol{v}^*, (v_2^*, d^* + v_2^*)\})$. Therefore, B^* dominates A^* as shown in Figure 2. Hence, by Lemma 8 (ii), $U(B) \geq U(A)$. It follows that $U(A) = U(B) = \max_{\boldsymbol{v}\in A^*} v_2 = \max_{\rho\in\Delta(A)} \sum_{f\in\mathscr{F}} \rho(f)U(f)$.

Case 2: Next, we consider the case in which $\delta > 0$. Define $U^* = \max_{\boldsymbol{v}\in A^*} \delta v_1 + (1-\delta)v_2$. We will show $U(A) = U^*$.

First, we show that $U^* \leq U(A)$. Let $\boldsymbol{v}^* \in \arg\max_{\boldsymbol{v}\in A^*} \delta v_1 + (1-\delta)v_2$. By Lemma 5, there exist $f, g \in \mathscr{F}$ such that $U(f) = v_2^* = U(g)$ and $\max_{\alpha\in[0,1]} U(\alpha f + (1-\alpha)g) - U(f) = v_1^* - v_2^*$. Let $B = \{f, g\}$. By Lemma 9, $U(B) = \delta v_1^* + (1-\delta)v_2^* = U^*$. Moreover, by Lemma 7 (i), $B^* = \mathrm{co}(\{(v_2^*, v_2^*), \boldsymbol{v}^*\})$. Since $B^* \subset A^*$ as shown in Figure 3, Lemma 8 (ii) shows that $U^* = U(B) \leq U(A)$.

In the following, we show $U(A) \leq U^*$. Define $\underline{U} = \min_{\boldsymbol{v}\in A^*} v_2$. Then, $U^* \geq \underline{U}$. By Lemma 5, there exist $f, g \in \mathscr{F}$ such that $f \sim g$, $U(f) = \underline{U}$, and $\max_{\alpha\in[0,1]} U(\alpha f + (1-\alpha)g) - U(f) = \frac{U^* - \underline{U}}{\delta}$. (Remember $\delta > 0$. So $\frac{U^* - \underline{U}}{\delta}$ is well defined.) Let $\alpha^* \in \arg\max_{\alpha\in[0,1]} U(\alpha f + (1-\alpha)g)$. Define $h^* = \alpha^* f + (1-\alpha^*)g$ and $x = U^* - U(h^*)$. Define $C = \{f, g\}$ and $D = \{h^* + x, f, g\}$. Then, by Lemma 9, $U(C) = \delta(\frac{U^* - \underline{U}}{\delta} + \underline{U}) + (1-\delta)\underline{U} = U^*$. By Lemma 7 (ii), $D^* = \mathrm{co}\big(\big\{(\underline{U}, \underline{U}), (\frac{U^* - \underline{U}}{\delta} + \underline{U}, \underline{U}), (U^*, U^*)\big\}\big)$.

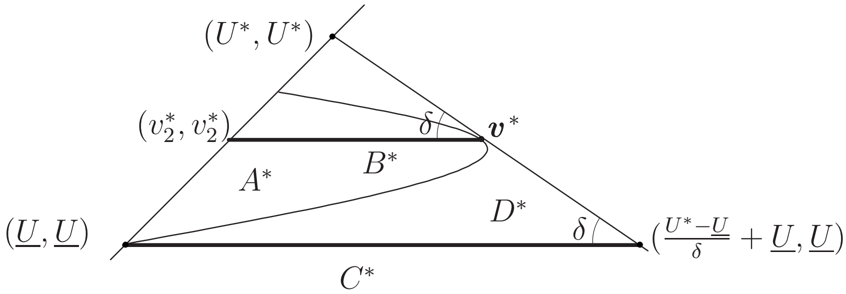

Figure 3: Sets A^*, B^*, C^*, and D^* in Case 2

By the definitions of U^* and $\underline{U}$, we obtain $A^* \subset D^*$ as shown in Figure 3. Then, by Lemma 8 (ii), $U(A) \le U(D)$. Hence, it suffices to show $U(D) = U^*$. By Lemma 6, the agent does not have a strict incentive to randomize $h^* + x$ with f and g. Since $U(\{f, g\}) = U^* = U(h^* + x)$, Weak Strategic Rationality shows $h^* + x \sim \{h^* + x, f, g\} \equiv D$, so that $U(D) = U^*$.

2.2 Proof of Necessity

The representation trivially satisfies Indifference. To show that the representation satisfies Weak Strategic Rationality, remember the definition of w: for all $\rho \in \Delta(\mathscr{F})$, $w(\rho) = \delta u(\sum_{f \in \mathscr{F}} \rho(f) f) + (1-\delta) \sum_{f \in \mathscr{F}} \rho(f) u(f)$. Then, for all $A \in \mathscr{A}$, $U(A) = \max_{\rho \in \Delta(A)} w(\rho)$.

Suppose that the agent does not have a strict incentive to randomize f with g and h and $f \succsim \{g, h\}$. Then, for any $\mu \in \Delta(\{f, g, h\})$, there exists $\rho \in \Delta(\{f, g, h\})$ such that (i) $w(\rho) \ge w(\mu)$ and (ii) $\rho_1 u(f) + (1-\rho_1) u(\frac{\rho_2}{1-\rho_1} g + \frac{\rho_3}{1-\rho_1} h) = u(\rho_1 f + \rho_2 g + \rho_3 h)$, where $\rho = \rho_1 f \oplus \rho_2 g + \rho_3 h$. So, we obtain

$$
\begin{aligned}
& U(\{f, g, h\}) \\
&= \max\nolimits_{\mu \in \Delta(\{f,g,h\})} w(\mu) \\
&= \max\nolimits_{\mu \in \Delta(\{f,g,h\})} \delta u(\mu_1 f + \mu_2 g + \mu_3 h) + (1-\delta)(\mu_1 u(f) + \mu_2 u(g) + \mu_3 u(h)) \\
&= \max\nolimits_{\rho \in \Delta(\{f,g,h\})} \delta\big(\rho_1 u(f) + (1-\rho_1) u(\tfrac{\rho_2}{1-\rho_1} g + \tfrac{\rho_3}{1-\rho_1} h)\big) + (1-\delta)(\rho_1 u(f) + \rho_2 u(g) + \rho_3 u(h)) \\
&= \max\nolimits_{\rho_1 \in [0,1]} \big[\rho_1 u(f) + (1-\rho_1) \max\nolimits_{\rho'_1 \in [0,1]} \delta u(\rho'_1 g + (1-\rho'_1) h) + (1-\delta)(\rho'_1 u(g) + (1-\rho'_1) u(h))\big] \\
&= \max\nolimits_{\rho_1 \in [0,1]} \big[\rho_1 u(f) + (1-\rho_1) \max\nolimits_{\rho' \in \Delta(\{g,h\})} w(\rho')\big].
\end{aligned}
$$

Since $f \succsim \{g, h\}$, then $u(f) \geq \max_{\rho' \in \Delta(\{g,h\})} w(\rho')$. It follows that $U(\{f, g, h\}) = u(f)$.

The uniqueness of δ can be proved exactly in the same way as in the proof of Remark 1 of the main paper.

3 Proofs of Lemmas 1 and 2 in Proof of Theorem 1

We provide the proofs of Lemmas 1 and 2.

Lemma 1: *There exists a nonempty, compact, and convex subset C of $\Delta(S)$ such that $\succsim$ on $\mathscr{F}$ is represented by $u(f) = \min_{p \in C} \sum_{s \in S} p(s) f(s)$.*

Proof of Lemma 1: Note that Certainty Set Independence implies Gilboa and Schmeidler's (1989) axiom: $f \succsim g \Leftrightarrow \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x$ for all $f, g \in \mathscr{F}$, $x \in \mathscr{U}$, and $\alpha \in (0,1)$. By identifying $\{f\}$ as f for all $f \in \mathscr{F}$, we can confirm that $\succsim$ satisfies all the conditions of Gilboa and Schmeidler's (1989) theorem. Hence, there exist a mixture linear function $\phi : \mathscr{U} \to \mathbb{R}$ and a nonempty, compact, and convex subset C of $\Delta(S)$ such that $\min_{p \in C} \sum_{s \in S} p(s)\phi(f(s)) \geq \min_{p \in C} \sum_{s \in S} p(s)\phi(g(s))$ if and only if $f \succsim g$. Define $\overline{u} = \max \mathscr{U}$ and $\underline{u} = \min \mathscr{U}$. Normalize ϕ by $\phi(\overline{u}) = \overline{u}$ and $\phi(\underline{u}) = \underline{u}$. Choose any $u \in \mathscr{U}$. Then, $u = \frac{u-\underline{u}}{\overline{u}-\underline{u}}\overline{u} + \big(1 - \frac{u-\underline{u}}{\overline{u}-\underline{u}}\big)\underline{u}$. Since ϕ is mixture linear, we obtain $\phi(u) = \phi\big(\frac{u-\underline{u}}{\overline{u}-\underline{u}}\overline{u} + \big(1 - \frac{u-\underline{u}}{\overline{u}-\underline{u}}\big)\underline{u}\big) = \frac{u-\underline{u}}{\overline{u}-\underline{u}}\overline{u} + \big(1 - \frac{u-\underline{u}}{\overline{u}-\underline{u}}\big)\underline{u} = u$.

Lemma 2: *There exists a function $U : \mathscr{A} \to \mathscr{U}$ such that (i) $U(A) \geq U(B) \Leftrightarrow A \succsim B$ and (ii) $U(f) = \min_{p \in C} \sum_{s \in S} p(s) f(s)$ for all $f \in \mathscr{F}$.*

Proof of Lemma 2: For all $f \in \mathscr{F}$, define $U(\{f\}) = u(f)$. Since C is compact, Berge's Maximum theorem shows that u is continuous. Fix $A \in \mathscr{A}$. Let $\overline{f} \in \arg\max_{f \in \mathrm{co}(A)} u(f)$ and $\underline{f} \in \arg\max_{f \in A} u(f)$. Since A is compact, Weierstrass's theorem shows that $\overline{f}$ and $\underline{f}$ exist. By Dominance, $\overline{f} \succsim A \succsim \underline{f}$. If $\overline{f} \sim A$ or $A \sim \underline{f}$, define $U(A) = u(\overline{f})$ or $U(A) = u(\underline{f})$, respectively. If $\overline{f} \succ A \succ \underline{f}$, then by Continuity, there exists $\alpha \in [0,1]$ such that $A \sim \alpha\overline{f} + (1-\alpha)\underline{f}$. By Monotonicity, such α is unique. Define $U(A) = u(\alpha\overline{f} + (1-\alpha)\underline{f})$. By the definition, U satisfies

(i) and (ii).

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