# Savage in the Market: Online Appendix (NOT FOR PUBLICATION)

Federico Echenique
Caltech

Kota Saito
Caltech

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This online appendix consists of five sections. In Section 1, we illustrate the use of SARSEU through a few simple theoretical exercises. In each case, the exercise is to present a well-known alternative to risk-averse SEU, and to show that data generated by these theories can violate SARSEU. In Section 2, we show that, in the $2 \times 2$ case, SARSEU is equivalent to Requirements (5) and (6) in the paper. In Section 3, we provide the details of the example of risk-averse probability sophisticated preferences that violate SARSEU. In Section 4, we present the relationship between SARSEU and results obtained in the revealed preference literature on *objective* expected utility, where the probability over states is assumed to be observable and given as a primitive. The point is that the axiom to characterize objective expected utility has a similar syntax to SARSEU, and that the differences between the two are instructive. In Section 5, we study the relationship between SARSEU and the axiomatization in Savage (1954). It is useful to see the role of SARSDU and SARSEU in ruling our violations of Savage's axioms.

# 1 Applications of SARSEU

We discuss, in turn, non-concave SEU and state-dependent utility. By showing that these models can generate datasets which violate SARSEU, we show that the models are in fact testable beyond risk-averse SEU. In other words, that non-concave SEU and state-dependent utility all have testable implications over and beyond those of risk-averse SEU. This point has been made by Bayer et al. (2012) for non-concave SEU and maxmin expected utility; but they use Afriat inequalities to this end, and here we seek to illustrate the use of SARSEU.

## 1.1 Non-concave SEU

The concavity of $u$ plays an important role in our characterization. This should not be surprising, as risk aversion has obvious economic meaning and content. There are, however, instances in revealed preference theory where concavity has no implications for a rational consumer. Afriat's theorem (Afriat (1967)) shows that concavity is not a testable property of a utility function. For the SEU model, concavity of $u$ is equivalent to the convexity of preferences over state-contingent bundles. So it is legitimate to ask about the testability of the concavity of $u$. In this section we show that indeed concavity is testable.

In the following, we will show an example of dataset generated from a non-concave SEU model that violates SARSEU.

Consider the following dataset:

$$p^{k_1} = (1, 2), x^{k_1} = (1, 2) \text{ and } p^{k_2} = (1.1, 2), x^{k_2} = (10, 1).$$

Note that

$$x^{k_1}_{s_2} > x^{k_2}_{s_2} \text{ and } x^{k_2}_{s_1} > x^{k_1}_{s_1},$$

while

$$\frac{p^{k_1}_{s_2}}{p^{k_2}_{s_2}} \frac{p^{k_2}_{s_1}}{x^{k_1}_{s_1}} = \frac{2}{2} \frac{1.1}{1} = 1.1 > 1,$$

so SARSEU is violated, and the dataset is not rationalizable by any concave utility and priors.

It is, however, rationalizable by the following non-concave SEU. Let $\mu = \left(\frac{1}{3}, \frac{2}{3}\right)$. Define

$$v(x) = \begin{cases} 1 & \text{if } x \leq 9, \\ 2 & \text{if } 9 < x \leq 10, \\ 1 & \text{if } x > 10. \end{cases}$$

Remember that $B(p, I) = \{x : \mathbf{R}^2_+ : p^k \cdot x \leq p^k \cdot x^k\}$ for all $p \in \mathbf{R}^S_{++}$ and $I \in \mathbf{R}_{++}$. Let $u(x) = \int_0^x v(s)ds$.

It is clear that $x^1$ is optimal for $\sum \mu_s u(x_s)$ in $B(p^1, p^1 \cdot x^1)$, as $v(x_{s_1}) = v(x_{s_2}) = 1$ for all $(x_{s_1}, x_{s_2}) \in B(p^1, p^1 \cdot x^1)$.

By the monotonicity of $u$, any maximum of $\sum \mu_s u(x_s)$ in $B(p^2, p^2 \cdot x^2)$ must lie on the budget line $p^{k_2}_{s_1} x_{s_1} + p^{k_2}_{s_2} x_{s_2} = 13$. Note that, on the budget line,

$$x_{s_2} = \frac{13 - 1.1 x_{s_1}}{2},$$

so $x_{s_2} \leq \frac{13}{2} < 9$ for $x_{s_1} \geq 0$. For all $x_{s_1} \geq 0$, define $f(x_{s_1}) = \mu_1 u(x_{s_1}) + \mu_2 u(x_{s_2}) = \frac{1}{3}\left[u(x_{s_1}) + 2u(\frac{13-1.1x_{s_1}}{2})\right]$. Then, $f'(x_{s_1}) = \frac{1}{3}\left[v(x_{s_1}) - 1.1\right]$ for $x_{s_1} \in [0, 13/1.1]$, as $v(\frac{13-1.1x_{s_1}}{2}) = 1$. Thus,

$$f'(x_{s_1}) = \begin{cases} \frac{-0.1}{3} & \text{if } x_{s_1} \leq 9, \\ \frac{0.9}{3} & \text{if } 9 < x_{s_1} \leq 10, \\ \frac{-0.1}{3} & \text{if } 10 < x_{s_1}. \end{cases}$$

So $f(x_{s_1})$ has two local maxima, $x_{s_1} = 0$ and $x_{s_1} = 10$. By a direct calculation, $f(0) = \frac{13}{6} = 2 + \frac{1}{6}$ and $f(10) = \frac{1}{3}(9 + 2) + \frac{2}{3}(\frac{13-1.1\times 10}{2}) = 3 + \frac{4}{3}$. Since $f(10) > f(0)$, it is indeed optimal to choose $x^2$ in $B(p^2, p^2 \cdot x^2)$.

### 1.2 State-dependent Utility

State-dependent utility is the model in which an agent seeks to maximize $\sum_{s \in S} u_s(x_s)$; where $u_s$ is a utility function over money for each state $s$. By means of Afriat inequalities, Varian (1983a) has proposed a characterization of additive linear model, which includes state-dependent utility model as a special case.

On the other hand, we have proposed a combinatorial axiom (i.e., SARSDU) for concave state-dependent utility, which is beyond Afriat inequalities. The axiom is weaker than SARSEU. Hence, the two models should be distinguishable. In the following, we propose an example of dataset that is consistent with SARSDU but not with SARSEU.

Assume $S = \{s_1, s_2\}$. Consider the following dataset:

$$p^{k_1} = (3, 2),\ p^{k_2} = (1, 1) \text{ and } x^{k_1} = (2, 1),\ x^{k_2} = (3, 4).$$

Choose strictly concave functions $u_{s_1}$ and $u_{s_2}$ such that

$$u'_{s_1}(2) = 3 > 1 = u'_{s_1}(3) \text{ and } u'_{s_2}(1) = 2 > 1 = u'_{s_2}(4).$$

Then

$$\frac{u'_{s_1}(2)}{u'_{s_2}(1)} = \frac{p^k_{s_1}}{p^k_{s_2}}, \quad \frac{u'_{s_1}(3)}{u'_{s_2}(4)} = \frac{p^{k2}_{s_1}}{p^{k2}_{s_2}},$$

so that the first-order conditions are satisfied.

The sequence $\{(x^{k_1}_{s_1}, x^{k_1}_{s_2}), (x^{k_2}_{s_2}, x^{k_2}_{s_1})\}$ satisfies the condition of SARSEU. However,

$$\frac{p^{k_1}_{s_1}}{p^{k_1}_{s_2}} \frac{p^{k_2}_{s_2}}{p^{k_2}_{s_1}} = \frac{3}{2} > 1.$$

This is a violation of SARSEU. Note also that this dataset violates Requirement (6).

## 2 Proof of Remark 1

In the paper, in Remark 1, we claimed that the following two requirements are equivalent to SARSEU in the $2 \times 2$ case. Here we prove this statement.

**Requirements:**

$$x^{k_1}_{s_1} > x^{k_2}_{s_1} \text{ and } x^{k_2}_{s_2} > x^{k_1}_{s_2} \Rightarrow \frac{p^{k_1}_{s_1}}{p^{k_2}_{s_1}} \frac{p^{k_2}_{s_2}}{p^{k_1}_{s_2}} \leq 1. \tag{5}$$

$$x^{k_1}_{s_1} > x^{k_1}_{s_2} \text{ and } x^{k_2}_{s_2} > x^{k_2}_{s_1} \Rightarrow \frac{p^{k_1}_{s_1}}{p^{k_1}_{s_2}} \frac{p^{k_2}_{s_2}}{p^{k_2}_{s_1}} \leq 1. \tag{6}$$

**Remark:** *In the $2 \times 2$ case, a dataset satisfies SARSEU if and only if it satisfies Requirement 5 and 6.*

*Proof.* It is easy to see that SARSEU implies the two requirements. To prove the reciprocal, we need a preliminary concept.

We say that a set $\sigma \equiv \{(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})\}_{i=1}^n$ of pairs that satisfies the three conditions in SARSEU is *minimal* if there exist no subset $(\sigma_i)_{i=1}^m$ of pairs such that (i) $\sigma_i$ satisfies the three conditions in SARSEU for each $i = 1, \ldots, m$ and (ii) $\sigma = \cup_{i=1}^n \sigma_i$.

**Step** 1: If a set $\{(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})\}_{i=1}^n$ of pairs is minimal and satisfies the three conditions, then it must take one of the following forms:

(a) $\sigma^a \equiv \{(x^{k_1}_{s_1}, x^{k_1}_{s_2}), (x^{k_2}_{s_2}, x^{k_2}_{s_1})\}$,

(b) $\sigma^b \equiv \{(x^{k_1}_{s_1}, x^{k_2}_{s_1}), (x^{k_2}_{s_2}, x^{k_1}_{s_2})\}$,

(c) $\sigma^c \equiv \{(x^k_s, x^{k'}_{s'}), (x^{k'}_{s'}, x^k_{s'}), (x^{k'}_{s'}, x^{k'}_s)\}$ for some $k, k'$ such that $k \neq k'$ and $s, s'$ such that $s \neq s'$,

(d) $\sigma^d \equiv \{(x^k_s, x^{k'}_{s'}), (x^{k'}_s, x^k_s), (x^k_s, x^k_{s'})\}$ for some $k, k'$ such that $k \neq k'$ and $s, s'$ such that $s \neq s'$.

**Proof of Step** 1: Fix a minimal set $\sigma = \{(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})\}_{i=1}^n$ of pairs that satisfies the three conditions.

**Case** 1: For all $i$, either $k_i = k'_i$ or $s_i = s'_i$ holds. It is easy to see that either $\sigma = \{(x^{k_1}_{s_1}, x^{k_1}_{s_2}), (x^{k_2}_{s_2}, x^{k_2}_{s_1})\}$ or $\sigma = \{(x^{k_1}_{s_1}, x^{k_2}_{s_1}), (x^{k_2}_{s_2}, x^{k_1}_{s_2})\}$. These correspond to Case (a) and (b).

**Case** 2: There exists $i$ such that neither $k_i = k'_i$ or $s_i = s'_i$ holds. Suppose that we have $(x^{k_1}_{s_1}, x^{k_2}_{s_2})$ in the pair without loss of generality.

We cannot have $(x^{k_2}_{s_2}, x^{k_1}_{s_1})$ in $\sigma$ because of condition (1). Since the sequence $\{(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})\}_{i=1}^n$ satisfies conditions (2) and (3), we must have two pairs such that $(x^{k_2}_s, x^{k_1}_s)$ and $(x^k_{s_2}, x^k_{s_1})$ for some $s \in \{s_1, s_2\}$ and $k \in \{k_1, k_2\}$ in the sequence.

Note that we cannot have $(x^{k_2}_{s_2}, x^{k_1}_{s_2})$ and $(x^{k_1}_{s_2}, x^{k_1}_{s_1})$ because these imply $x^{k_2}_{s_2} > x^{k_1}_{s_2} > x^{k_1}_{s_1}$, which would contradict that $(x^{k_1}_{s_1}, x^{k_2}_{s_2})$ in $\sigma$ satisfies condition (1).

Note also that we cannot have $(x^{k_2}_{s_1}, x^{k_1}_{s_1})$ and $(x^{k_2}_{s_2}, x^{k_2}_{s_1})$ because these imply $x^{k_2}_{s_2} > x^{k_2}_{s_1} > x^{k_1}_{s_1}$. This again contradicts that $(x^{k_1}_{s_1}, x^{k_2}_{s_2})$ in $\sigma$ satisfies condition (1). It follows that we must be in one of the following two cases.

**Subcase** 2.1: The pairs $(x_{s_2}^{k_2}, x_{s_2}^{k_1})$ and $(x_{s_2}^{k_2}, x_{s_1}^{k_2})$ are in $\sigma$. Then $\{(x_{s_1}^{k_1}, x_{s_2}^{k_2}), (x_{s_2}^{k_2}, x_{s_2}^{k_1}), (x_{s_2}^{k_2}, x_{s_1}^{k_2})\}$ satisfies the three conditions. Since $\sigma$ is minimal, it must hold that $\sigma = \{(x_{s_1}^{k_1}, x_{s_2}^{k_2}), (x_{s_2}^{k_2}, x_{s_2}^{k_1}), (x_{s_2}^{k_2}, x_{s_1}^{k_2})\}$. This corresponds to Case (c).

**Subcase** 2.2: The pairs $(x_{s_1}^{k_2}, x_{s_1}^{k_1})$ and $(x_{s_2}^{k_1}, x_{s_1}^{k_1})$ are in $\sigma$. In this case, again by the minimality of $\sigma$, it must hold that $\sigma = \{(x_{s_1}^{k_1}, x_{s_2}^{k_2}), (x_{s_1}^{k_2}, x_{s_1}^{k_1}), (x_{s_2}^{k_1}, x_{s_1}^{k_1})\}$. This corresponds to Case (d).

Note that we have exhausted all cases. □

For any set $\sigma \equiv \{(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})\}_{i=1}^{n}$ of pairs that satisfies the three conditions in SARSEU, define

$$f(\sigma) = \prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}}.$$

**Step** 2: $f(\sigma^t) \leq 1$ for each $t \in \{a, b, c, d\}$.

**Proof of Step** 2: By Requirement (5),

$$f(\sigma^a) = \frac{p_{s_1}^{k_1}}{p_{s_1}^{k_2}} \frac{p_{s_2}^{k_2}}{p_{s_2}^{k_1}} \leq 1.$$

By Requirement (6),

$$f(\sigma^b) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}} \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1.$$

To show that $f(\sigma^c) \leq 1$, assume without loss of generality that $\sigma^c = \{(x_{s_1}^{k_1}, x_{s_2}^{k_2}), (x_{s_2}^{k_2}, x_{s_2}^{k_1}), (x_{s_2}^{k_2}, x_{s_1}^{k_2})\}$. In this case, it must hold that $x_{s_2}^{k_2} > x_{s_1}^{k_2}$ and $x_{s_1}^{k_1} > x_{s_2}^{k_1}$. Hence, by Requirement (6),

$$\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}} \leq 1.$$

So

$$f(\sigma^c) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_2}}{p_{s_2}^{k_1}} \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} = \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}} \leq 1.$$

To show that $f(\sigma^d) \leq 1$, assume without loss of generality that $\sigma^d = \{(x_{s_1}^{k_1}, x_{s_2}^{k_2}), (x_{s_1}^{k_2}, x_{s_1}^{k_1}), (x_{s_2}^{k_1}, x_{s_1}^{k_1})\}$. In this case, it must hold that $x_{s_1}^{k_2} > x_{s_2}^{k_2}$ and $x_{s_2}^{k_1} > x_{s_1}^{k_1}$. Hence, by Requirement (6),

$$\frac{p_{s_1}^{k_2}}{p_{s_1}^{k_2}} \frac{p_{s_2}^{k_1}}{p_{s_1}^{k_1}} \leq 1.$$

So

$$f(\sigma^d) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_1}^{k_2}}{p_{s_1}^{k_1}} \frac{p_{s_2}^{k_1}}{p_{s_1}^{k_1}} = \frac{p_{s_1}^{k_2}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_1}}{p_{s_1}^{k_1}} \leq 1.$$

□

Now by using Steps 1 and 2, we can prove the remark. Choose a sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ that satisfies the three conditions in SARSEU. Let $\sigma = \{(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})\}^n_{i=1}$. For each $t$, there exists set of index $M(t)$ such that $\sigma = (\cup_{i\in M(a)}\sigma^a)\cup(\cup_{i\in M(b)}\sigma^b)\cup(\cup_{i\in M(c)}\sigma^c)\cup(\cup_{i\in M(d)}\sigma^d)$.

$$\prod_{i=1}^{n} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} = f(\sigma) = \Big(\prod_{i\in M(a)} f(\sigma^a)\Big)\dots\Big(\prod_{i\in M(d)} f(\sigma^d)\Big) \leq 1.$$

□

# 3 Probabilistic Sophistication

We present the detailed arguments behind the example in Section 5 of the paper. The example has a dataset that is generated by risk-averse probability sophisticated preferences, but which violates SARSEU.

In the example, we have $S = \{s_1, s_2\}$ and $x^1 = (2, 2)$, $p^1 = (1, 2)$, $x^2 = (8, 0)$, and $p^2 = (1, 1)$. In the following, we define the function $V$ that represents the probabilistically sophisticated preferences. Fix $\mu \in \Delta_{++}$ with $\mu_{s_1} = \mu_{s_2} = 1/2$. Any vector $x \in \mathbf{R}^2_+$ induces the probability distribution on $\mathbf{R}_+$ given by $x_1$ with probability $1/2$ and $x_2$ with probability $1/2$. Let $\Pi$ be the set of all probability measures on $\mathbf{R}_+$ for which the support is finite and has cardinality smaller than or equal to 2.

We shall define a function $V : \Pi \to \mathbf{R}$ that represents probabilistically sophisticated preferences. We use $h : \{(x_1, x_2) \in \mathbf{R}^2_+ : x_1 \geq x_2\} \to \mathbf{R}$ with the property that

$$h(x_1, x_2) \leq h\Big(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\Big),$$

and then define $V(\pi) = h(\bar{x}^\pi, \underline{x}^\pi)$, where $\underline{x}^\pi$ is the smallest point in the support of $\pi$, and $\bar{x}^\pi$ is the largest.

Let $C_1$ be the set of vectors $x$ for which $x_1 + x_2 \geq 8$, $C_3$ be the set of vectors $x$ for which $x_1 + 2x_2 \leq 6$, and $C_2$ be the complement of $C_1 \cup C_3$ in $\{(x_1, x_2) \in \mathbf{R}^2_+ : x_1 \geq x_2\}$. It is easy to see that these sets are pairwise disjoint.

For $\sigma \in (6, 8)$, let $l^\sigma \subset \mathbf{R}^2_+$ be the set of vectors on the line segment (or the convex hull of) between $(\sigma, 0)$ and $(\sigma - 4, \sigma - 4)$. We have the following basic properties of $\sigma$.

**Lemma 1.** *Suppose that* $\sigma \in (6, 8)$.

*1.* $(x_1, x_2) \in l^\sigma$ *if and only if*

$$x_2 = \Big(\frac{\sigma - 4}{4}\Big)(\sigma - x_1),$$

*2.* $l^\sigma \subset C_2$*,*

*3. For each* $x \in C_2$ *there is a unique* $\sigma \in (6, 8)$ *with* $x \in l^\sigma$*.*

*Proof.* The proof of statement 1 is a direct calculation. For statement 2 note that $\sigma < 8$ and $\sigma - 4 + \sigma - 4 < 8$. The function $(x_1, x_2) \mapsto x_1 + x_2$ is linear and is therefore maximized on $l^\sigma$ at an extreme. Since this function is smaller than 8 on both extremes, it is smaller than 8 over all $l^\sigma$ and therefore $l^\sigma$ does not intersect $C_1$. Similarly, it does not intersect $C_3$ by the linearity of $x_1 + 2x_2$ and checking the extremes.

For statement 3, fix $\sigma \in (6, 8)$. Consider the function $f^\sigma(x_1, x_2) = (\sigma - 4)x_1 + 4x_2$. One can verify that $f^\sigma(\sigma, 0) = \sigma(\sigma - 4) = f^\sigma(\sigma - 4, \sigma - 4)$. Then $f^\sigma(x_1, x_2) = \sigma(\sigma - 4)$ for all $(x_1, x_2) \in l^\sigma$, as $f^\sigma$ is linear. Consider $\sigma' \neq \sigma$. Say $\sigma' > \sigma$. Then the minimum of $f^\sigma$ over $l^{\sigma'}$ is obtained at an extreme point of $l^{\sigma'}$, again by linearity. Now, $\sigma' > \sigma$ implies that $f^\sigma(\sigma, 0) < f^\sigma(\sigma', 0)$ and $f^\sigma(\sigma - 4, \sigma - 4) < f^\sigma(\sigma' - 4, \sigma' - 4)$. Hence $f^\sigma(x) < f^\sigma(x')$ for all $x \in l^\sigma$ and $x' \in l^{\sigma'}$. Thus if $x \in l^\sigma$ then $x \notin l^{\sigma'}$.

We complete the proof of statement 3 by showing that for all $x \in C_2$ there is $\sigma$ with $x \in l^\sigma$. Let $x \in C_2$. Consider the quadratic equation $\sigma^2 - (4 + x_1)\sigma + 4(x_1 - x_2) = 0$, derived from the identity in statement 1 of this lemma. By solving this equation explicitly and choosing the larger solution, we obtain

$$\sigma = \frac{4 + x_1 + \sqrt{-8x_1 + x_1^2 + 16(1 + x_2)}}{2}.$$

By a direct calculation, it can be shown that

$$[\sigma > 6 \Leftrightarrow 6 < x_1 + 2x_2] \text{ and } [\sigma < 8 \Leftrightarrow x_1 + 2x_2 > 6].$$

Since $x \in C^2$, we obtain $\sigma \in (6, 8)$.

□

Define the function $h : \{(x_1, x_2) \in \mathbf{R}^2_+ : x_1 \geq x_2\} \to \mathbf{R}$ as follows:

$$h(x) = \begin{cases} \frac{1}{2}(x_1 + x_2) & \text{if } x \in C_1, \\ \frac{1}{3}(x_1 + 2x_2) & \text{if } x \in C_3, \\ \sigma(x_1, x_2) - 4 & \text{if } x \in C_2, \end{cases}$$

where $\sigma(x_1, x_2)$ is the unique $\sigma \in (6, 8)$ with $x \in l^\sigma$.

**Lemma 2.** *If $x \in C_3$, $x' \in C_2$, and $x'' \in C_1$, then $h(x) < h(x') < h(x'')$.*

*Proof.* Let $\sigma \in \mathbf{R}_+$ be such that $x' \in l^\sigma$. We must have $8 > \sigma > 6$, so

$$h(x) = \frac{1}{3}(x_1 + 2x_2) \leq \frac{1}{3}6 < \sigma - 4 = h(x'),$$

as $x_1 + 2x_2 \leq 6$ and $\sigma > 6$; and

$$h(x') = \sigma - 4 < 4 \leq \frac{1}{2}(x_1'' + x_2'') = h(x''),$$

as $8 > \sigma$ and $x_1'' + x_2'' \geq 8$. □

**Lemma 3.** *For any $(x_1, x_2) \in \{(x_1, x_2) \in \mathbf{R}^2_+ : x_1 \geq x_2\}$,*

$$h(x_1, x_2) \leq h\Big(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\Big).$$

*Proof.* First, let $x \in C_3$. If $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}) \notin C_3$ the result follows from Lemma 2. So suppose that $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}) \in C_3$. Then

$$h\Big(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\Big) = \frac{1}{3}\Big(\frac{x_1 + x_2}{2} + x_1 + x_2\Big) \geq \frac{1}{3}(x_1 + 2x_2) = h(x_1, x_2),$$

where the inequality follows from $x_1 \geq x_2$.

Second, suppose that $x \in C_1$. Then $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}) \in C_1$ and it is immediate that $h(x) = h(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2})$.

Thirdly, suppose that $x \in C_2$. It is easy to see that $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}) \notin C_3$ (as $\frac{x_1+x_2}{2} + 2\frac{x_1+x_2}{2} = x_1 + x_2 + \frac{x_1+x_2}{2} \geq x_1 + 2x_2 > 6$), and the result follows from Lemma 2 when $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}) \in C_1$, so consider the case when $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}) \in C_2$.

Let $\sigma \in (6, 8)$ be such that $x \in l^\sigma$. Consider the function $(x_1', x_2') \mapsto x_1' + x_2'$ for $(x_1', x_2') \in l^\sigma$. Note that $(x_1', x_2') \in l^\sigma$ means that

$$x_1' + x_2' = x_1' + \frac{\sigma - 4}{4}(\sigma - x_1') = \left(1 - \frac{\sigma - 4}{4}\right)x_1 + \frac{\sigma - 4}{4}\sigma,$$

which is monotone increasing in $x_1'$, as $\sigma < 8$. But $(x_1, x_2) \in l^\sigma$ implies that $x_1 \geq \sigma - 4$, hence $(\sigma - 4) + (\sigma - 4) \leq x_1 + x_2$. Thus,

$$h(x) = \sigma - 4 \leq \frac{x_1 + x_2}{2} = h\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right).$$

□

**Lemma 4.** *For all $x, x' \in \{(x_1, x_2) \in \mathbf{R}^2_+ : x_1 \geq x_2\}$, if $x'_2 \leq x_2$ and $x'_1 \leq x_1$, then $h(x'_1, x'_2) \leq h(x_1, x_2)$. If $x'_2 < x_2$ or $x'_1 < x_1$, furthermore, then $h(x'_1, x'_2) < h(x_1, x_2)$.*

*Proof.* First $x \in C_1$. Then $x' \in C_1$, and there is nothing more to prove. In second place suppose that $x \in C_2$. Then we cannot have $x' \in C_3$ as $x_1 + 2x_2 \leq x'_1 + 2x'_2$. The result follows from Lemma 2 if $x' \in C_1$, so focus on the case when $x' \in C_2$. It suffices to show that $\sigma(x_1, x_2)$ is strictly increasing both in $x_1$ and in $x_2$. As shown in the proof of Lemma 3,

$$\sigma(x_1, x_2) = \frac{4 + x_1 + \sqrt{-8x_1 + x_1^2 + 16(1 + x_2)}}{2}.$$

By a direct calculation,

$$\begin{aligned} \frac{\partial \sigma}{\partial x_1} &= \frac{1}{2}\Big(1 + \frac{-4 + x_1}{\sqrt{-8x_1 + x_1^2 + 16(1 + x_2)}}\Big), \\ \frac{\partial \sigma}{\partial x_2} &= \frac{4}{\sqrt{-8x_1 + x_1^2 + 16(1 + x_2)}}. \end{aligned}$$

Hence, $\frac{\partial \sigma}{\partial x_2} > 0$. Since $\sqrt{-8x_1 + x_1^2 + 16(1 + x_2)} > -4 + x_1$, we also obtain $\frac{\partial \sigma}{\partial x_1} > 0$.

Finally, when $x \in C_3$ the conclusion either follows from Lemma 2 or from the monotonicity of $x_1 + 2x_2$. □

As noted in the main paper, we define $V : \Pi \to \mathbf{R}$ by $V(\pi) = h(\bar{x}^\pi, \underline{x}^\pi)$, where $\underline{x}^\pi$ is the smallest point in the support of $\pi$, and $\bar{x}^\pi$ is the largest. Recall that $h$ is defined on $\{(x_1, x_2) \in \mathbf{R}^2_+ : x_1 \geq x_2\}$. So, the definition of $V$ shows that the preferences represented by $V$ are symmetric across 45 degree line. Write $\pi \prec \pi'$ if $\pi'$ strictly first-order stochastically dominates $\pi$. Let $F_\pi$ and $F_{\pi'}$ be the CDFs of $\pi$ and $\pi'$, respectively.

**Lemma 5.** *$\pi \prec \pi'$ implies that $V(\pi) < V(\pi')$.*

*Proof.* Assume $\pi \prec \pi'$. By Lemma 4, it suffices to show that $\underline{x}^\pi \leq \underline{x}^{\pi'}$ and $\bar{x}^\pi \leq \bar{x}^{\pi'}$ and that at least one of the inequalities is strict. Suppose that $\underline{x}^\pi > \underline{x}^{\pi'}$ or $\bar{x}^\pi > \bar{x}^{\pi'}$. By choosing $x$ such that $\underline{x}^{\pi'} < x < \min\{\bar{x}^{\pi'}, \underline{x}^\pi\}$, we have $F_{\pi'}(x) = 1/2 > 0 = F_\pi(x)$. This contradicts that $\pi \prec \pi'$. Hence, $\underline{x}^\pi \leq \underline{x}^{\pi'}$. In the same way, we obtain $\bar{x}^\pi \leq \bar{x}^{\pi'}$. Moreover, since $\pi \neq \pi'$, we obtain $\underline{x}^\pi < \underline{x}^{\pi'}$ or $\bar{x}^\pi < \bar{x}^{\pi'}$. □

For any $\pi \in \Pi$, let $e(\pi)$ be the degenerate lottery that yields the expected value of $\pi$ (recall that $\pi$ has finite support) with probability 1. The following result is immediate from Lemma 3.

**Lemma 6.** $V(\pi) \leq V(e(\pi))$

Lemma 5 establishes that $V$ represents probabilistically sophisticated preferences. Lemma 6 says that the preferences are also risk-averse. We now proceed to verify that the dataset defined at the outset is rationalizable an agent with utility function $V$. We write $\pi_x \in \Pi$ for the probability measure induced by $x \in \mathbf{R}^2_+$.

For observation 1, the budget set is

$$B(p^1, p^1 \cdot x^1) = \{x \in \mathbf{R}^2_+ : x_1 + 2x_2 \leq 6\}.$$

For all $x \in B(p^1, p^1 \cdot x^1)$, it is clear that $\max\{x_1, x_2\} + 2\min\{x_1, x_2\} \leq x_1 + 2x_2 \leq 6$, and hence that

$$V(\pi_x) = \frac{1}{3}(\max\{x_1, x_2\} + 2\min\{x_1, x_2\}) \leq 2 = V(\pi_{x^1})$$

For observation 2, the budget set is

$$B(p^2, p^2 \cdot x^2) = \{x \in \mathbf{R}^2_+ : x_1 + x_2 \leq 8\}.$$

Note that $h(x) \leq 2$ for all $x \in C_3$ and $h(x) \leq 4$ for all $x \in C_2$. Since $B(p^2, p^2 \cdot x^2) \cap C_1$ consists of vectors for which $x_1 + x_2 = 8$, and $h(x_1, x_2) = 4$ for those vectors, it follows that $V(\pi_x) \leq V(\pi_{x^2})$ for all $x \in B(p^2, p^2 \cdot x^2)$.

# 4 Objective Expected Utility

In this section, we present the relationship between our main theorem and results in Green and Srivastava (1986), Varian (1983b), and Kubler et al. (2014). These authors discuss a setting where an objective probability $\mu \in \Delta_{++}$ is given. Given the objective probability $\mu$, they seek to understand when there is a utility function for which the observed purchases maximize expected utility.

We show that we can write a version of our SARSEU that uses "risk neutral" prices in place of regular prices. We show that this modified axiom characterizes the objective expected utility theory. Our modified SARSEU is therefore equivalent to the conditions studied by Green and Srivastava (1986) and Varian (1983b), and to the axiom in Kubler et al. (2014).

It is worth emphasizing that Kubler et al. (2014) allows $\mu$ to depend on $k$, so that the agent may use a different prior when faced with different optimization problems. In our subjective probability setup this would make no sense because everything is rationalizable by suitably choosing priors in each optimization problem. Here we are being consistent with the rest of the paper in assuming a fixed prior through all observations, but the result can be relaxed to fit a variable-prior setup.

**Definition 1.** *A dataset* $(x^k, p^k)_{k=1}^K$ *is* objective expected utility (OEU) rational *if there is a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* $k$*,*

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x_s^k).$$

In the papers cited above, a crucial aspect of the dataset are the price-probability ratios, or "risk neutral prices," defined as follows: for $k \in K$ and $s \in S$

$$\rho_s^k = \frac{p_s^k}{\mu_s}.$$

A natural modification of SARSEU using the objective probability $\mu$ is as follows:

**Strong Axiom of Revealed Objective Expected Utility (SAROEU):** *For any sequence of pairs* $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ *in which*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ *for all* $i$*;*
2. *each* $k$ *appears in* $k_i$ *(on the left of the pair) the same number of times it appears in* $k_i'$ *(on the right):*

*The product of price-probability ratios satisfies that*

$$\prod_{i=1}^n \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} \leq 1.$$

The prior $\mu$ is observable, so we do not need the requirement on $s$ in SARSEU. Instead, SAROEU restricts the products of price-probability ratios, and not the product of price ratios.

The notion of dataset in Kubler et al. (2014) is the same as in our paper. Kubler et al. (2014) investigate the case of strict concave utility, while we have focused on weak concavity. A modification of Kubler et. al's axiom that allows for weak concavity is as follows:[1]

[1]SAREU and Kubler et. al's axiom are different only in one point: their axiom requires $\prod_{i=1}^m \left( \max_{s,s' : x_s^{k(i)} > x_{s'}^{k(i+1)}} \frac{\rho_s^{k(i)}}{\rho_{s'}^{k(i+1)}} \right) < 1$.

**Strong Axiom of Revealed Expected Utility (SAREU):** *For all* $m \geq 1$ *and sequences* $k(1), \dots, k(m) \in K$,

$$\prod_{i=1}^{m} \Big( \max_{s,s':x_s^{k(i)} > x_{s'}^{k(i+1)}} \frac{\rho_s^{k(i)}}{\rho_{s'}^{k(i+1)}} \Big) \leq 1.$$

It is easy to modify the argument in Kubler et al. (2014) to show the equivalence of a dataset being OEU-rational, satisfying the conditions in Green and Srivastava (1986) and Varian (1983b).

**Proposition 7.** *A dataset is OEU-rational if and only if it satisfies SAROEU.*

This result implies that SAROEU, SAREU, and the conditions in Green and Srivastava (1986) and Varian (1983b) are equivalent.

*Proof.* Using the result of Kubler et al. (2014), we prove the result by establishing the equivalence between SAROEU and SAREU.

Suppose that the dataset $(x^k, p^k)_{k=1}^K$ satisfies SAROEU. Suppose, by way of contradiction, that SAREU is violated. Then there exist $m \geq 1$ and $k_1, \dots, k_m \in K$ such that $\prod_{i=1}^m \big( \max_{s,s':x_s^{k_i} > x_{s'}^{k_{i+1}}} (\rho_s^{k_i}/\rho_{s'}^{k_{i+1}}) \big) > 1$. If $m = 1$ it clearly contradicts SAROEU.

In the following we will consider the case where $m > 1$. Then, there exists a sequence $(x_{s_i^*}^{k_i^*}, x_{s_i'^*}^{k_{i+1}^*})_{i=1}^m$ with $k_{m+1}^* = k_1^*$ such that $\prod_{i=1}^m (\rho_{s_i^*}^{k_i^*}/\rho_{s_i'^*}^{k_{i+1}^*}) > 1$. Since the sequence satisfies the conditions in SAROEU, this contradicts SAROEU.

Now, we establish that SAREU implies SAROEU. Choose a sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ of pairs in which $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$; each $k$ appears in $k_i$ (on the left of the pair) as many as in $k_i'$ (on the right). If $n = 1$, we have that $k_i = k_i' = k$. Consider the sequence $k(1) = k = k(2)$. Then SAREU implies that $\rho_s^k/\rho_{s'}^k \leq 1$, as desired.

Now, consider the case in which $n \geq 2$.

**Step** 1: There exists a collection of cycles such that each cycle $(k(i))_{i=1}^{2m}$ satisfies (i) $x_{s(i)}^{k(i)} > x_{s'(i+1)}^{k(i+1)}$ for $i = 1, 3, \dots, 2m-1$ and (ii) $k(2m) = k(1)$.

**Proof of Step** 1: First consider the pair $(x_{s_1}^{k_1}, x_{s_1'}^{k_1'})$. Let $k(1) = k_1$ and $k(2) = k_1'$. Since each $k$ appears as $k_i$ as many times as $k_i'$, there exists a pair $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})$ with $k_i = k(2)$. Let $k(3) = k_i$ and $k(4) = k_i'$. If $k(4) = k(1)$, then we have a cycle in $k$. Otherwise, for the same reason as was mentioned above, there is a $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})$ with $k_i = k(4)$. We can now let $k(5) = k_i$ and $k(6) = k_i'$. If $k(6) = k(1)$, then we again have a cycle. Since the number of data that appear in the sequence we started from is finite, we must eventually close a cycle. Each time we find a cycle, we can start the procedure from any remaining pair $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})$ in the dataset. Since each $k$ appears in $k_i$ the same number of times it appears in $k'_i$, we must exhaust all pairs after finding a finite collection of cycles.

**Step** 2: $\prod_{i=1}^{n} \frac{\rho^{k_i}_{s_i}}{\rho^{k'_i}_{s'_i}} \leq 1$.

**Proof of Step** 2: For each cycle $(k(i))_{i=1}^{2m}$, we have that

$$\prod_{i=1}^{m/2} \frac{\rho^{k(2i-1)}_{s(2i-1)}}{\rho^{k(2i)}_{s'(2i)}} \leq \prod_{i=1}^{m/2} \Big( \max_{s,s':x^{k(2i-1)}_s > x^{k(2i)}_{s'}} \Big( \frac{\rho^{k(2i-1)}_s}{\rho^{k(2i)}_{s'}} \Big) \Big) \leq 1,$$

as

$$\frac{\rho^{k(2i-1)}_{s(2i-1)}}{\rho^{k(2i)}_{s'(2i)}} \leq \max_{s,s':x^{k(2i-1)}_s > x^{k(2i)}_{s'}} \Big( \frac{\rho^{k(2i-1)}_s}{\rho^{k(2i)}_{s'}} \Big).$$

Then, since the product over each cycle does not exceed 1, the product of the cycles satisfies that:

$$\prod_{i=1}^{n} \frac{\rho^{k_i}_{s_i}}{\rho^{k'_i}_{s'_i}} \leq 1.$$

□

# 5 Relationship between SARSEU and Savage's (1954) axioms

In this section, we study the relationship between SARSEU and Savage's (1954) axioms. Savage's axiomatization involves seven axioms, labeled P1-P7. We show that SARSEU implies Savage's axioms, except for P1 and P6: P1 requires preference to be a weak order, which does not make sense for our primitive. P6 requires the set of states to be infinite.

Specifically, it is interesting to disentangle the role of SARSDU (Section 4 of the paper) and SARSEU in ruling out violations of Savage's axioms. It turns out that a violation of P2 or P7 will imply a violation of SARSDU, the axiom behind a state-dependent representation. This makes sense, as P2 and P7 are essentially separability assumption. A violation of P4 has a different structure, and we show that it violates SARSEU (actually it violates Requirement (6)). Finally, P3 and P5 cannot be violated in our setup.

In this section, we use the following notations: For any $A \subset S$ and $x \in \mathbf{R}^S_+$ and $p \in \mathbf{R}^S_+$, $x_A$ denotes the vector in $\mathbf{R}^A_+$ obtained by restricting $s \mapsto x_s$ to $A$; similarly, $p_A$ denotes the vector in $\mathbf{R}^A_+$ obtained by restricting $s \mapsto p_s$ to $A$.

Recall that Savage's primitive is a complete preference relation over acts. In contrast, our primitive is a dataset $(x^k, p^k)_{k=1}^K$. To relate the two models, we define a revealed preference relation from the dataset $(x^k, p^k)_{k=1}^K$ and investigate when it satisfies Savage's axioms.

**Definition 2.** *For any $x, y \in \mathbf{R}_+^S$,*
*(i) $x \succeq y$ if there exists $k \in K$ such that $x = x^k$ and $p^k \cdot x \geq p^k \cdot y$;*
*(ii)$x \succ y$ if there exists $k \in K$ such that $x = x^k$ and $p^k \cdot x > p^k \cdot y$.* [2]

There is one basic problem: Savage's primitive is a complete preference relation over acts, but a dataset will contain much less information than a preference relation over $\mathbf{R}_+^S$. The revealed preference relation is going to be incomplete: many acts in $\mathbf{R}_+^S$ will not be comparable. Such incompleteness gives rise to trivial violations of Savage's axioms, as his axioms were formulated for complete preferences. For example, one of Savage's axiom is as follows:

**Axiom (P2).** *Let $x, y, x', y' \in \mathbf{R}_+^S$ and $A \subset S$ such that $x_A = x'_A$ and $y_A = y'_A$ and $x_{A^c} = y_{A^c}$ and $x'_{A^c} = y'_{A^c}$. Then $x \succeq y$ if and only if $x' \succeq y'$.*

The revealed preference relation violates P2 when only one of $x, y$ and $x', y'$ are comparable. This is not a particularly interesting violation of Savage's axioms. A more meaningful exercise is to show how a violation of Savage's axioms that is not due to incompleteness implies a violation of SARSEU.

**Definition 3.** *Let $\succeq$ be the revealed preference relation defined from $(x^k, p^k)_{k=1}^K$ by Definition 2. Then we say that the dataset violates P2 if there is $x, y, x', y' \in \mathbf{R}_+^S$ and $A \subset S$ as in the statement of P2 for which $x \succeq y$ and $y' \succ x'$; or $y \succeq x$ and $x' \succeq y'$.*

**Proposition 8.** *If the dataset violates P2, then it violates SARSDU.*

*Proof.* For a subset $A$ of $S$ and a dataset $(x^k, p^k) \in \mathbf{R}_+^S \times \mathbf{R}_{++}^S$, we consider $(x_A^k, p_A^k) \in \mathbf{R}_+^A \times \mathbf{R}_{++}^A$. This defines a dataset $(x_A^k, p_A^k)_{k=1}^K$ on a restricted domain with $A$ (instead of $S$). On this restricted domain, we can define WARP and SARSDU in the same way as we defined in Section 4 of the paper. SARSDU implies WARP on this restricted domain.

Suppose that the dataset $(x^k, p^k)_{k=1}^K$ violates P2. Then by definition of $\succeq$, and the fact that $x_{A^c} = y_{A^c}$ and $x'_{A^c} = y'_{A^c}$, the dataset $(x_A^k, p_A^k)_{k=1}^K$ violates WARP. Then $(x_A^k, p_A^k)_{k=1}^K$ violates SARSDU, which implies that $(x^k, p^k)_{k=1}^K$ violates SARSDU. □

[2] It is worth emphasizing that this definition already has separability built in, which goes a long way to satisfying Savage's axioms.

We shall use the following notation. We use $1_A$ to denote the indicator vector for $A \subset S$ in $\mathbf{R}^S_+$; and for a scalar $x \in \mathbf{R}_+$, $x1_A$ denotes the vector in $\mathbf{R}^A_+$ with $x$ in all its entries (the constant vector $x$).

**Axiom (P4).** *Suppose $A, B \subset S$; $x > y$, $x' > y'$. Then, $(x1_A, y1_{A^c}) \succ (x1_B, y1_{B^c})$ if and only if $(x'1_A, y'1_{A^c}) \succ (x'1_B, y'1_{B^c})$.*

**Definition 4.** *Let $\succeq$ be the revealed preference relation defined from $(x^k, p^k)_{k=1}^K$ by Definition 2. Then we say that the dataset violates P4 if there is $A, B \subset S$ and scalars $x$, $x'$, $y$ and $y'$ as in the statement of P4 for which $(x1_A, y1_{A^c}) \succ (x1_B, y1_{B^c})$ and $(x'1_B, y'1_{B^c}) \succeq (x'1_A, y'1_{A^c})$, or $(x1_B, y1_{B^c}) \succeq (x1_A, y1_{A^c})$ and $(x'1_A, y'1_{A^c}) \succ (x'1_B, y'1_{B^c})$.*

**Proposition 9.** *If a dataset violates P4, then it violates SARSEU.*

*Proof.* Without loss of generality, we can assume that $\sum_{s\in S} p^k_s = 1$ for all $k$. The reason is that we can normalize prices to add up to 1 without affecting the validity of SARSEU.

Let $A, B \subset S$, and let $x$, $x'$, $y$ and $y'$ be scalars as in the statement of P4, such that $(x1_A, y1_{A^c}) \succ (x1_B, y1_{B^c})$ and $(x'1_B, y'1_{B^c}) \succeq (x'1_A, y'1_{A^c})$. Suppose, towards a contradiction, that the dataset satisfies SARSEU.

First, $(x1_A, y1_{A^c}) \succ (x1_B, y1_{B^c})$ means that there is an observation $k$ for which $x^k = (x1_A, y1_{A^c})$ and

$$p^k \cdot x^k = p^k \cdot (x1_A, y1_{A^c}) > p^k \cdot (x1_B, y1_{B^c}), \tag{7}$$

while $(x'1_B, y'1_{B^c}) \succeq (x'1_A, y'1_{A^c})$ means that there is an observation $k'$ such that

$$p^{k'} \cdot x^{k'} = p^{k'} \cdot (x'1_B, y'1_{B^c}) \geq p^{k'} \cdot (x'1_A, y'1_{A^c}). \tag{8}$$

Secondly, Equation (7) implies that

$$x \sum_{s\in A} p^k_s + y \left(1 - \sum_{s\in A} p^k_s\right) > x \sum_{s\in B} p^k_s + y \left(1 - \sum_{s\in B} p^k_s\right),$$

and therefore that $\sum_{s\in A} p^k_s > \sum_{s\in B} p^k_s$, as $x > y$. Similarly, Equation (8) and $x' > y'$ implies that $\sum_{s\in B} p^{k'}_s \geq \sum_{s\in A} p^{k'}_s$. Hence

$$\sum_{s\in A\setminus B} p^k_s > \sum_{s\in B\setminus A} p^k_s \text{ and } \sum_{s\in B\setminus A} p^{k'}_s \geq \sum_{s\in A\setminus B} p^{k'}_s \tag{9}$$

Thirdly, for any $s \in A \setminus B$ and any $s' \in B \setminus A$ we have that $x = x^k_s > x^k_{s'} = y$, and $x' = x^{k'}_{s'} > x^{k'}_s = y'$. Hence, SARSEU implies that

$$\frac{p^k_s}{p^k_{s'}} \frac{p^{k'}_{s'}}{p^{k'}_s} \leq 1.^3$$

[3]Observe that here we are essentially using Requirement (6).

Thus, for any $s' \in B \setminus A$,

$$p_{s'}^{k'} \sum_{s \in A \setminus B} p_s^k \leq p_{s'}^k \sum_{s \in A \setminus B} p_s^{k'},$$

which implies that

$$\sum_{s' \in B \setminus A} p_{s'}^{k'} \sum_{s \in A \setminus B} p_s^k \leq \sum_{s' \in B \setminus A} p_{s'}^k \sum_{s \in A \setminus B} p_s^{k'},$$

a contradiction with (9). □

We now discuss P3 and P7 (P5 is a non-triviality axiom that is always satisfied in our setup). This requires some preliminary definitions.

**Definition 5.** *For any $A \subset S$ and $x_A, y_A \in \mathbf{R}^A$,*
*(i) $x_A \succeq_A y_A$ if there exist $z, w \in \mathbf{R}^S$ such that $z_A = x_A$ and $w_A = y_A$ and $z_{A^c} = w_{A^c}$, $z \succeq w$.*
*(ii) $x_A \succ_A y_A$ if there exist $z, w \in \mathbf{R}^S$ such that $z_A = x_A$ and $w_A = y_A$ and $z_{A^c} = w_{A^c}$, $z \succ w$.*

**Definition 6.** *$A \subset S$ is* null *if for any $x, y \in \mathbf{R}_+^S$ such that $x_{A^c} = y_{A^c}$, it is false that $x \succ y$.*

**Axiom (P3).** *Suppose that $A$ is not null. Then, $x1_A \succ_A y1_A$ if and only if $x > y$.*

**Axiom (P7).** *(i) $x_s 1_A \succ_A y_A$ for all $s \in A$ implies $x_A \succ_A y_A$; (ii) $y_A \succ_A x_s 1_A$ for all $s \in A$ implies $y_A \succ_A x_A$.*

**Definition 7.** *Let $\succeq$ be the revealed preference relation defined from $(x^k, p^k)_{k=1}^K$ by Definition 2. Then we say that the dataset violates*

1. *P3 if there is non null $A \subseteq S$, $x, y \in \mathbf{R}_+$, and $z \in \mathbf{R}_+^{A^c}$, for which $(x1_A, z) \succ (y1_A, z)$ and $y \geq x$, or $(y1_A, z) \succeq (x1_A, z)$ and $x > y$.*

2. *P7 if there is a nonempty $A \subseteq S$ and $x, y \in \mathbf{R}_+^S$ such that one of the following is true*

   *(a) $y_A \succeq_A x_A$ while $x_s 1_A \succ_A y_A$ for all $s \in A$;*

   *(b) $x_A \succeq_A y_A$ while $y_A \succ_A x_s 1_A$ for all $s \in A$;*

**Proposition 10.** *(i) No dataset can violate P3; (ii) if a dataset violates P7, then it violates SARSDU.*

*Proof.* Fix a dataset that violates P3. Let $A$, $x$, $y$ and $z$ be as in the definition of a violation of P3. Suppose that $(x1_A, z) \succ (y1_A, z)$ and $y \geq x$. Then there is an observation $x^k = (x1_A, z)$ with

$$x \sum_{s \in A} p_s^k + \sum_{s \in A^c} p_s^k z_s = p^k \cdot (x1_A, z) > p^k \cdot (y1_A, z) = y \sum_{s \in A} p_s^k + \sum_{s \in A^c} p_s^k z_s.$$

Hence, $x > y$, as $\sum_{s \in A} p_s^k > 0$. This contradicts that the dataset violates P3.

Suppose that $y_A \succeq_A x_A$ while $x_s 1_A \succ_A y_A$ given $A$ for all $s \in A$. Let $k$ be such that $y_A = x_A^k$. For $s \in A$, let $k_s$ be such that $x_s 1_A = x_A^{k_s}$. $y_A \succeq_A x_A$ implies $p_A^k \cdot x_A^k \geq p_A^k \cdot x_A$. For all $s \in A$, $x_s 1_A \succ_A y_A$ implies $p_A^{k_s} \cdot x_A^{k_s} > p_A^{k_s} \cdot x_A^k$.

Let $s^*$ be such that $x_{s^*} \leq x_s$ for all $s \in A$. Then $p_A^k \cdot x_A^k \geq p_A^k \cdot x_A$ implies that $p_A^k \cdot x_A^k \geq p_A^k \cdot x_A^{k_{s^*}}$.

Now, $p_A^{k_{s^*}} \cdot x_A^{k_{s^*}} > p_A^{k_{s^*}} \cdot x_A^k$ implies that the dataset $(x_A^k, p_A^k)_{k=1}^K$ violates WARP on the restricted domain. So the dataset $(x_A^k, p_A^k)_{k=1}^K$ must violate SARSDU on the restricted domain. Hence, the dataset $(x_A^k, p_A^k)_{k=1}^K$ must violate SARSDU in the original domain. □

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