ONLINE APPENDIX: TESTABLE IMPLICATIONS OF TRANSLATION INVARIANCE AND HOMOTHETICITY: VARIATIONAL, MAXMIN, CARA AND CRRA PREFERENCES

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In this online appendix, we provide the complete proof of Theorem 4. We prove some lemmas.

Lemma 1: A dataset D is maxmin rationalizable if and only if there are v_s^k, λ^k, $s = 1, 2$, $k = 1, \dots, K$, and $\bar{\pi}, \underline{\pi} \geq 0$ with $\bar{\pi} \geq \underline{\pi} > 0$, such that: for all k such that $x_s^k \neq x_{s'}^k$,

$$\pi v_1^k = \lambda^k p_1^k$$
$$v_2^k = \lambda^k p_2^k,$$

where $\pi = \bar{\pi}$ when $x_1^k < x_2^k$ and $\pi = \underline{\pi}$ when $x_1^k > x_2^k$; for all k such that $x_s^k = x_{s'}^k$

$$\overline{\pi} v_1^k \geq \lambda^k p_1^k$$
$$\underline{\pi} v_1^k \leq \lambda^k p_1^k$$
$$v_2^k = \lambda^k p_2^k.$$

The numbers also satisfy that $v_s^k \leq v_{s'}^{k'}$ when $x_s^k > x_{s'}^{k'}$.

Proof: To prove sufficiency, let v_s^k, λ^k, $s = 1, 2$, $k = 1, \dots, K$, and $\bar{\pi}, \underline{\pi} \geq 0$ with $\bar{\pi} \geq \underline{\pi}$ be as in the statement of the lemma. Define $\bar{\mu}, \underline{\mu} \in \Delta(S)$ as follows. Let $\bar{\mu}_1 = \bar{\pi}/(1+\bar{\pi})$, $\bar{\mu}_2 = 1/(1+\bar{\pi})$, and $\underline{\mu}_1 = \underline{\pi}/(1+\underline{\pi})$, $\underline{\mu}_2 = 1/(1+\underline{\pi})$. Since

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$\bar{\pi} \geq \underline{\pi}$,

$$\bar{\mu}_1 \geq \underline{\mu}_1 \text{ and } \bar{\mu}_2 \leq \underline{\mu}_2.$$

Define $\theta^k = \lambda^k/(1+\bar{\pi})$ if $x_1^k < x_2^k$ and $\theta^k = \lambda^k/(1+\underline{\pi})$ if $x_1^k > x_2^k$. Then we have that $\mu_s v_s^k = \theta^k p_s^k$, with $\mu_s = \bar{\mu}_s$ when $x_1^k < x_2^k$; and $\mu_s = \underline{\mu}_s$ when $x_1^k > x_2^k$.

Now consider k such that $x_1^k = x_2^k$. By the assumption, there exists π^k such that $\overline{\pi} \geq \pi^k \geq \underline{\pi}$ such that $\pi v_1^k = \lambda^k p_1^k$. Let $\mu_1^k = \pi/(1+\pi)$, $\mu_2^k = 1/(1+\pi)$, and $\theta^k = \lambda^k/(1+\pi)$. Since $\overline{\pi} \geq \pi \geq \underline{\pi}$,

$$\overline{\mu}_1 \equiv \frac{\overline{\pi}}{1+\overline{\pi}} \geq \frac{\pi^k}{1+\pi^k} \geq \frac{\underline{\pi}}{1+\underline{\pi}} \equiv \underline{\mu}_1.$$

Hence, there exists $\alpha^k \in [0,1]$, $\mu_1^k = \alpha^k \overline{\mu}_1 + (1-\alpha^k)\underline{\mu}_1$. Then, $\mu_2^k = 1 - \mu_1^k = \alpha^k(1-\overline{\mu}_1) + (1-\alpha^k)(1-\underline{\mu}_1) = \alpha^k \overline{\mu}_2 + (1-\alpha^k)\underline{\mu}_2$.

Given the numbers v_s^k it is now routine to define a correspondence ρ such that if $x \leq x'$, $y \in \rho(x)$ and $y' \in \rho(x')$ then $y \geq y' > 0$, and with $\rho(x_s^k) \ni v_s^k$. This gives a concave and increasing function u with $\partial u(c) = \rho(x)$. So $\frac{\theta^k p_s^k}{\mu_s} \in \partial u(x_s^k)$ for all s and k such that $x_1^k \neq x_2^k$. Moreover, for all k such that $x_1^k = x_2^k$

$$(\theta^k p_1^k, \theta^k p_2^k) \in \text{co}\left\{ \left(\overline{\mu}_1 \partial u(x_1^k), \overline{\mu}_2 \partial u(x_2^k)\right), \left(\underline{\mu}_1 \partial u(x_1^k), \underline{\mu}_2 \partial u(x_2^k)\right) \right\}.$$

Hence the first and second order conditions are satisfied for maxmin rationalization. We omit the proof of necessity.

We will define matrices A, B, E such that there exist numbers $\{v_s^k\}$, $\{\lambda^k\}$, $\bar{\pi}$, $\underline{\pi}$ satisfying the conditions in Lemma 1 if and only if there exists a solution x to the system of inequalities $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$.

Let A be a matrix with $2K + 2 + K + 1$ columns. The first $2K$ columns are labeled with a different pair (k, s). The next 2 columns are labeled $\bar{\pi}$ and $\underline{\pi}$. The next K columns are labeled with a $k \in \{1, \ldots, K\}$. Finally the last column is labeled p.

For each $(k, 2)$ with $k \in K_0$, A has a row with all zero entries with the following exception. It has a 1 in the column labeled (k, s), among the first group of $2K$ columns. It has a -1 in the column labeled k. In the column labeled p it has $-\log(p_s^k)$.

For each (k, s) with $k \in K_1$, A has a row with all zero entries with the following exception. It has a 1 in the column labeled (k, s), among the first group of $2K$ columns. It has a -1 in the column labeled k. In the column labeled p it has $-\log(p_s^k)$. Finally, it has a 1 in the column labeled $\bar{\pi}$ if and only if if $s = 1$. For each (k, s) with $k \in K_2$, A has a row defined as above. The only difference is that when $s = 1$ then it has a 1 in the column labeled $\underline{\pi}$ instead of having a 1 in the column labeled $\bar{\pi}$.

Let B be a matrix with the same number of columns as A. The columns of B are labeled like those of A. For each $(k, 1)$ with $k \in K_0$, B has two rows. In the first row, B has a row with all zero entries with the following exception. It has a 1 in the column labeled (k, s), among the first group of $2K$ columns. It has a 1 in the column labeled $\bar{\pi}$. It has a -1 in the column labeled k. In the column labeled p it has $-\log(p_s^k)$. In the second row, B has a row with all zero entries with the following exception. It has a -1 in the column labeled (k, s), among the first group of $2K$ columns. It has a -1 in the column labeled $\underline{\pi}$. It has a 1 in the column labeled k. In the column labeled p it has $\log(p_s^k)$.

In addition, B has a row for each pair $(x_s^k, x_{s'}^{k'})$ with $x_s^k > x_{s'}^{k'}$. The row for $x_s^k > x_{s'}^{k'}$ has all zeroes except for a 1 in column (k', s') and a -1 in column (k, s). Finally, B has one more row. This row as a 1 in the column for $\bar{\pi}$ and a -1 in the column for $\underline{\pi}$.

Let E be a matrix with the same number of columns as A, labeled as above, and a single row. The row has all zeroes except for a 1 in column p.

By Lemma 1, there is no rationalizing maxmin preference if and only if there is no solution to the system of inequalities $A \cdot x = 0$, $B \cdot x \geq 0$, and $E \cdot x > 0$.

Suppose that all $\log(p_s^k)$ are rational numbers. We shall use the following version of the Theorem of the Alternative, which can be found as Theorem 1.6.1 in Stoer and Witzgall (1970).

Lemma*: Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong to a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.

(1) There is $u \in \mathbf{F}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, $E \cdot u \gg 0$.
(2) There is $\eta \in \mathbf{F}^m$, $\theta \in \mathbf{F}^l$, and $\gamma \in \mathbf{F}^r$ such that $\eta \cdot A + \theta \cdot B + \gamma \cdot E = 0$; $\theta \geq 0$ and $\gamma > 0$.

Then the non-existence of a solution to the system $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$ is equivalent to the existence of integer vectors η, θ and γ such that $\theta \geq 0$, $\gamma > 0$, and $\eta \cdot A + \theta \cdot B + \gamma E = 0$.

For a matrix D with $2K + 2 + K + 1$ columns, let D_1 denote the submatrix corresponding to the first $2K$ columns, D_2 correspond to the next 2, D_3 to the next K, and D_4 to the last column. Note that, by construction of A, B and E, $\eta \cdot A + \theta \cdot B + \gamma E = 0$ implies that $\eta \cdot A_1 + \theta \cdot B_1 = 0$, $\eta \cdot A_2 + \theta \cdot B_2 = 0$, $\eta \cdot A_3 + \theta \cdot B_3 = 0$, $\eta \cdot A_4 + \theta \cdot B_4 + \gamma = 0$. In fact, we can without loss assume that η, θ and γ take values of -1, 0 or 1. (This assumption is without loss because we can replace each row of matrices A, B and E with as many copies as indicated by the corresponding vector η, θ or γ.)

From the existence of such vectors it follows that we can obtain a sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ with $x^{k_i}_{s_i} > x^{k'_i}_{s'_i}$. The source of each pair $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})$ is that the column (k_i, s_i) of A is multiplied by $\eta_{(k_i,s_i)} > 0$ and the column (k'_i, s'_i) of A is multiplied by $\eta_{(k'_i,s'_i)} < 0$. The vector η must then have $\eta_{(k_i,s_i)} > 0$ and $\eta_{(k'_i,s'_i)} > 0$, with a -1 in the first column and a 1 in the second.

We shall prove that the sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ satisfies the properties stated in the axiom.

Firstly, $\eta \cdot A_3 + \theta \cdot B_3 = 0$ means that for each k, the number of is for which $k = k_i$ equals the number of is for which $k = k'_i$.

Secondly, $\eta \cdot A_2 + \theta \cdot B_2 = 0$ implies that:

$$\sum_{k \in K_1} \eta_{(k,1)} + \sum_{k \in K_0} \theta_{(k,1)} + \theta_{\overline{\pi} \geq \underline{\pi}} = 0$$

$$\sum_{k \in K_2} \eta_{(k,1)} - \sum_{k \in K_0} \theta'_{(k,1)} - \theta_{\overline{\pi} \geq \underline{\pi}} = 0,$$

where $\theta_{\overline{\pi} \geq \underline{\pi}}$ is the nonnegative weight on the row associated with $\overline{\pi} \geq \underline{\pi}$; $\theta_{(k,1)}$ and $\theta'_{(k,1)}$ are the nonnegative weights on the two rows associated with $(k, 1)$ with $k \in K_0$. ($\theta_{(k,1)}$ and $\theta'_{(k,1)}$ corresponds to the first row and the second row, respectively). Note that

$$\begin{aligned}
\sum_{k\in K_1}\eta_{(k,1)} &= \#\{i : k_i \in K_1, s=1\} - \#\{i : k'_i \in K_1, s=1\} \\
&\equiv \#I_{1,1} - \#I'_{1,1}, \\
\sum_{k\in K_2}\eta_{(k,1)} &= \#\{i : k_i \in K_2, s=1\} - \#\{i : k'_i \in K_2, s=1\} \\
&\equiv \#I_{2,1} - \#I'_{2,1}, \\
\sum_{k\in K_0}\theta_{(k,1)} &= \#\{i : k_i \in K_0, s=1\} \equiv \#I_{0,1}, \\
\sum_{k\in K_0}\theta'_{(k,1)} &= \#\{i : k'_i \in K_0, s=1\} \equiv \#I'_{0,1}.
\end{aligned}$$

Hence,

$$\#I_{1,1} - \#I'_{1,1} + \#I_{0,1} = \#I'_{2,1} - \#I_{2,1} + \#I'_{0,1} \leq 0.$$

Therefore the sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ satisfies the second property stated in the axiom. Finally, since $\eta \cdot A_4 + \theta \cdot B_4 + \gamma = 0$,

$$\begin{aligned}
0 &> -\gamma \\
&= \eta \cdot A_4 + \theta \cdot B_4 \\
&= \sum_{(k,s)\in(K_0,2)\cup(K_1\cup K_2,1)} \eta_{(k,s)}(-\log p^k_s) \\
&\quad + \sum_{k\in K_0}\theta_{(k,1)}(-\log p^k_1) + \sum_{k\in K_0}\theta'_{(k,1)}\log p^k_1 \\
&= \sum_{i=1}^n \log\frac{p^{k'_i}_{s'_i}}{p^{k_i}_{s_i}}
\end{aligned}$$

Hence

$$\prod_{i=1}^n \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} > 1.$$

The above proof assumes that the log of prices is rational. The proof of the theorem follows along the same lines as Echenique and Saito (2015). Specifically, we have shown the following.

Lemma 2: If $\{(x^k, p^k)\}$ is a dataset satisfying SARMEU, in which $\log p^k \in \mathbf{Q}$ for all k, then the dataset is maxmin rationalizable.

One can then prove the following

Lemma 3: If $\{(x^k, p^k)\}$ is a dataset that satisfies SARMEU, and $\varepsilon > 0$ then there is a collection of prices $\{q^k\}$ such that $\log q^k \in \mathbf{Q}$, $\|p^k - q^k\| < \varepsilon$, and the dataset $\{(x^k, q^k)\}$ satisfies SARMEU.

The proof of Lemma 3 is exactly the same as in Echenique and Saito (2015).

Lemma 2 establishes the result in datasets in which the log of prices is rational. Consider an arbitrary data set $\{(x^k, p^k)\}$, with prices that may not be rational.

Suppose towards a contradiction that the dataset satisfies SARMEU, but that it is not maxmin rational. Specifically then, by Lemma 1, suppose that there is no solution to the system $A \cdot x = 0$, $B \cdot x \geq 0$ and $E \cdot x > 0$. Then by Lemma* there are real vectors η, θ and γ such that $\theta \geq 0$, $\gamma > 0$, and $\eta \cdot A + \theta \cdot B + \gamma E = 0$.

Let $\{q^k\}$ be vectors of prices such that the dataset $\{(x^k, q^k)\}$ satisfies SARMEU and $\log q^k_s \in \mathbf{Q}$ for all k and s. (Such $\{q^k\}$ exists by Lemma 3.) Furthermore, the prices q^k can be chosen arbitrarily close to p^k. Construct matrices A', B', and E' from this dataset in the same way as A, B, and E above. Note that only the prices are different in $\{(x^k, q^k)\}$ compared to $\{(x^k, p^k)\}$. So $E' = E$, $B'_i = B_i$ and $A'_i = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A'_4 and B'_4 may be different from A_4 and B_4, respectively.

By Lemma 3, we can choose prices q^k such that $|\eta \cdot A'_4 + \theta \cdot B'_4 - (\eta \cdot A_4 + \theta \cdot B_4)| < \gamma/2$. We have shown that $\eta \cdot A_4 + \theta \cdot B_4 = -\gamma$, so the choice of prices q^k guarantees that $\eta \cdot A'_4 + \theta \cdot B'_4 < 0$. Let $\gamma' = -\eta \cdot A'_4 - \theta \cdot B'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \gamma' E_i = 0$ for $i = 1, 2, 3$. Hence

$$\eta \cdot A'_4 + \theta \cdot B'_4 + \gamma' E_4 = \eta \cdot A'_4 + \theta \cdot B_4 + \gamma' = 0.$$

We also have that $\eta \geq 0$ and $\gamma' > 0$. Therefore θ, η, and γ' exhibit a solution to the dual system for dataset $\{(x^k, q^k)\}$, a contradiction with Lemma 2.

References

Echenique, F. and K. Saito (2015): "Savage in the Market," *Econometrica*, 83, 1467–1495.

Stoer, J. and C. Witzgall (1970): *Convexity and Optimization in Finite Dimensions*, Springer-Verlag Berlin.