Testable Implications of Translation Invariance and Homotheticity: Variational, Maxmin, CARA and CRRA preferences

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We describe the observable content of some of the most widely used models of decision under uncertainty: models of translation invariant preferences. In particular, we characterize the models of variational, maxmin, CARA and CRRA utilities. In each case we present a revealed preference axiom that is satisfied by a dataset if and only if the dataset is consistent with the corresponding utility representation. We test our axioms using data from an experiment on financial decisions.

INTRODUCTION

This paper is an investigation of the testable implications of models of decision under uncertainty. We carry out this investigation in financial markets, one of the most common environments in which human subjects face uncertainty.

Risk is uncertainty which can be objectively quantified probabilistically. A gambler in a casino faces risk: he may calculate the probability that a roulette wheel stops on the number 7, or that a die lands on 5. Most scientists, in contrast, face the more general concept of uncertainty, and study subjects who face uncertainty. Scientists conduct or analyze experiments with outcomes that they do not know, and for which no probabilities are objectively given.

Of course, the scientist or the subject may have a subjective judgement of how likely different events are. Such judgements may even have a probabilistic expression, but the uncertainty is not resolved by means of a mechanical device for which probabilities can be objectively calculated. Moreover, this fact may cause subjects to display *uncertainty aversion,* a tendency to prefer risky bets over uncertain ones. Uncertainty aversion was famously documented by David Ellsberg [7], and the theories we treat in our paper are in part designed to describe uncertainty aversion.

In uncertain situations, human subjects choose among *uncertain prospects.* These are functions specifying an "outcome" for each element of a given set of "states of the world." Think of an insurance contract that pays off a given sum only if some accident occurs. The set of states of the world is the binary set that codifies whether an accident has occurred, and the outcome is the payoff. In financial markets, the uncertain prospects correspond to financial assets, while the state of the worlds describe the relevant economic fundamentals, and the outcomes monetary payoffs.[1]

A long tradition in decision theory develops models of how humans make decisions under uncertainty. A crucial idea in this development is that of *translation invariance.* Translation invariance means that if two uncertain prospects are transformed in the same way, by adding to each prospect a given, fixed, monetary payment, then the subject's preference between the two prospects should be preserved. For example, if the subject prefers insurance contract A over B, then the preference should be maintained after the price of each insurance contract has been raised by the same amount. A related idea is *homotheticity* where scaling the payoffs of the two contracts should not affect how they are ranked. Translation invariance and homotheticity give rise to different theories of decision under uncertainty.

Theories demand to be tested, and our contribution lies in working out the testable implications of theories of homothetic and translation invariant behavior under uncertainty. We focus on financial markets because these are some of the most familiar and common uncertain environments for human subjects. If one is to test a theory, it makes sense to study it in the subjects' most familiar environments. It is plausible that agents do not know how to behave in an artificial environment, but that they have learned how to deal with uncertainty in familiar environments. For human subjects, few uncertain environments are as familiar as financial markets. Most existing experimental environments are artificial: they involve human subjects choosing among bets on extractions of colored balls from urns of uncertain composition (Ellsberg's thought experiments are the best known of these; [7]). Our contribution is instead to focus on designs based on financial markets.

Our main results characterize the financial datasets that are consistent with the theories. Given is a finite collection of data on purchases of financial assets. The question is when are such data consistent with a theory of choice under uncertainty. We provide answers for some of the most commonly encountered theories, those based on translation invariance and homotheticity.

We show that our results are applicable to the analysis and design of experiments by using a recent experiment by Hey and Pace: [11]. Hey and Pace have subjects decide on purchases of financial assets. We use the data they collect to test

Significance

The paper uncovers the empirical content of many behavioral models of decisions under uncertainty. Studies of global financial markets often ignore a very important piece of the puzzle: individual behavior. We provide tests (and as such, predictions) about how individuals behave when facing uncertainty, such as that faced in financial or asset markets. Behavior at the individual level must be understood before the behavior of the economy at large can even begin to be understood.

Reserved for Publication Footnotes

[1] In probability theory, an uncertain prospect together with an underlying probability over the states of the world is termed a *random variable.*

for consistency with maxmin expected utility, a theory of decision under uncertainty based on translation invariance and homotheticity. The conclusion of our analysis is that Hey and Pace's data reject the maxmin theory. The finding is preliminary, and meant mainly as an illustration of our methods, but if confirmed it would mean that some of the best known theories of choice under uncertainty, theories that are thought of as weak, and accommodating of diverse behavioral and psychological phenomena, do not in fact stand up to empirical scrutiny on data from financial experiments.

The theories covered by our results include risk neutral variational preferences [13], risk neutral maxmin preferences [8], and subjective expected utility preferences with constant absolute risk aversion: so-called CARA preferences. Analogously to the CARA case, we also work out the testable implications of subjective expected utility preferences with constant relative risk aversion: so-called CRRA preferences (these form the "homothetic" class alluded to in the title). The theories have been used for different purposes. Variational and maxmin preferences are the most commonly-used models of uncertainty aversion [8, 5, 13, 14]. They are also used to capture model robustness [10]. CARA and CRRA preferences are extremely common in applied work in macroeconomics and finance, among other fields.

[9], [20], and [12] carry out similar exercises to ours, also focusing on financial market experiments, but in a context of risk, not uncertainty. The closest papers to ours are [6],[2] and [15]. [6] studies the case of subjective expected utility; it does not address the more general theories studied here, and that have been proposed to address the empirical shortcomings of subjective expected utility. [2] and [15] treat some of the same theories as we do, but give a characterization in terms of the solution of a system of inequalities. We give a revealed preference axiom (a characterization that references only observable data) that has to be satisfied for the data to be rationalizable. It can be written in the UNCAF form, which is the kind of axiom that characterizes the empirical content of a theory [4]. A system of (nonlinear) inequalities may not give an economic interpretation to the characterization, and it may not be computationally feasible.[2]

DEFINITIONS

Let S be a finite set of states of the world. An *act* is a function from S into $\mathbf{R}$; $\mathbf{R}^S$ is the set of acts. An act can be interpreted as a state-contingent monetary payment. Define $\|x\|_1 = \sum_s x_s$. $\Delta(S)$ represents the set of probability distributions on S, *i.e.* $\Delta(S) = \{\pi \in \mathbf{R}^S_+ : \sum_s \pi_s = 1\}$.

A *preference relation* on $\mathbf{R}^S$ is a complete and transitive binary relation $\succeq$; we denote by $\succ$ the strict part of $\succeq$. A function $u : \mathbf{R}^S \to \mathbf{R}$ defines a preference relation $\succeq$ by $x \succeq y$ if and only if $u(x) \geq u(y)$. We say that u *represents* $\succeq$, or that it is a utility function for $\succeq$. A preference relation $\succeq$ on $\mathbf{R}^S$ is *locally nonsatiated* if for every x and every $\epsilon > 0$ there is y such that $\|x - y\| < \epsilon$ and $y \succ x$.

PREFERENCES, UTILITIES, AND DATA

A *data set* D is a finite collection $\{(p^k, x^k)\}_{k=1}^K$, where each $p^k \in \mathbf{R}^S_{++}$ is a vector of strictly positive (so-called Arrow-Debreu) prices, and each $x^k \in \mathbf{R}^S$ is an act. The interpretation of a dataset is that each pair (p^k, x^k) consists of an act x^k chosen from the budget $\{x \in \mathbf{R}^S : p^k \cdot x \leq p^k \cdot x^k\}$ of affordable acts. Such data sets are common in financial markets experiments: [1, 2, 11].

A data set $\{(p^k, x^k)\}_{k=1}^K$ is *rationalizable* by a preference relation $\succeq$ if $x^k \succeq x$ whenever $p^k \cdot x^k \geq p^k \cdot x$. So a data set is rationalizable by a preference relation when the choices in the dataset would have been optimal for that preference relation. A data set $\{(p^k, x^k)\}_{k=1}^K$ is *rationalizable* by a utility function u if it is rationalizable by the preference relation represented by u. So a data set is rationalizable by a utility function when the choices in the dataset would have maximized that utility function in the relevant budget set.

A preference relation $\succeq$ is *translation invariant* if for all $x, y \in \mathbf{R}^S$ and all $c \in \mathbf{R}$, we have $x \succeq y$ if and only if $x + (c, \ldots, c) \succeq y + (c, \ldots, c)$.

A preference relation $\succeq$ is *homothetic* if for all $x, y \in \mathbf{R}^S$ and all $\alpha > 0$, we have $x \succeq y$ if and only if $\alpha x \succeq \alpha y$.

A preference relation $\succeq$ is a *risk-neutral variational preference* if there is a convex and lower semicontinuous function $c : \Delta(S) \to \mathbf{R} \cup \{+\infty\}$ such that the function

$$\inf_{\pi \in \Delta(S)} \pi \cdot x + c(\pi)$$

represents $\succeq$. If a data set is rationalizable by a risk-neutral variational preference relation, we will say that the dataset set is *risk-neutral variational-rationalizable.*

A special case of variational preference is maxmin: A preference relation is *risk-neutral maxmin* if there is a closed and convex set $\Pi \subseteq \Delta(S)$ such that the utility function

$$\inf_{\pi \in \Pi} \pi \cdot x$$

represents $\succeq$. If a data set is rationalizable by a risk neutral maxmin preference relation, we will say that the dataset set is *risk-neutral maxmin-rationalizable.* More generally, a preference relation is *risk averse maxmin* if there is a closed and convex set $\Pi \subseteq \Delta(S)$, where for each $\pi \in \Pi$ and each $s \in S$, $\pi_s > 0$, and a concave utility $u : \mathbf{R}^S \to \mathbf{R}$ such that the utility function

$$\inf_{\pi \in \Pi} \sum_{s=1,2} \pi_s u(x_s)$$

represents $\succeq$. If a data set is rationalizable by a maxmin preference relation, we will say that the dataset set is *maxmin-rationalizable.*

A utility $u : \mathbf{R}^S \to \mathbf{R}$ is *constant absolute risk aversion (CARA)* if there is $\alpha > 0$ and $\pi \in \Delta(S)$ for which for all $s \in S$, $\pi_s > 0$, and

$$u(x) = \sum_{s \in S} \pi_s \left(-\exp(-\alpha x)\right).$$

Note that CARA is a special case of subjective expected utility.[3]

A utility $u : \mathbf{R}^S \to \mathbf{R}$ is *constant relative risk aversion (CRRA)* if there is $\alpha \in (0, 1)$ and $\pi \in \Delta(S)$ for which for all $s \in S$, $\pi_s > 0$, and

$$u(x) = \sum_{s \in S} \pi_s \left(\frac{x^{1-\alpha}}{1-\alpha}\right).$$

If a data set is rationalizable by a CARA (CRRA) utility, we will say that the dataset set is *CARA (CRRA) rationalizable.*

[2] The paper [2] is a case in point, where the solution to the system of inequalities is implemented by a grid search. A conclusive test is not possible since they results depend on the assumed granularity of the grid.

[3] In fact it is also a special case of a risk neutral variational preference, a fact exploited by [19].

VARIATIONAL AND MAXMIN PREFERENCES

We present the results on variational and maxmin rationalizability as Theorems 1 and 2. These models satisfy the hypothesis that for any x, y, $x \sim y \implies \frac{1}{2}x + \frac{1}{2}y \succeq y$. This hypothesis is known as *convexity* of preference. Convexity is related to *uncertainty aversion* in the sense of [8]. In fact, given the assumptions of monotonicity found in that paper, together with the assumption that the preference is risk neutral (*i.e.* lotteries are evaluated according to their expected value), it is equivalent to uncertainty aversion. Uncertainty aversion is the idea that an agent dislikes uncertainty, and suffers from his ignorance of the possible probability distribution that governs outcomes.

One important conclusion that emerges from our analysis is that convexity is not testable with market data. This therefore means that under the maintained hypothesis of risk neutrality (and monotonicity), uncertainty aversion cannot be detected with financial data.

Theorem 1: The following statements are equivalent:

1. Dataset D is rationalizable by a locally nonsatiated, translation invariant preference.
2. Dataset D is rationalizable by a continuous, strictly increasing, concave utility function satisfying the property $u(x + (c, \dots, c)) = u(x) + c$.
3. Dataset D is risk-neutral variational-rationalizable.
4. For every $l = 1, \dots, M$, and every sequence $\{k_l\} \subseteq \{1, \dots, K\}$,

$$\sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) \geq 0,$$

where addition is modulo M, as usual.

Note that the equivalence between (2) and (3) is due to [13].

Remark: The fact that (1) implies (2) and (3) implies that if data are rationalizable by a translation invariant preference, they are also rationalizable by a risk-neutral variational preference (which automatically satisfies convexity).

Remark: The preceding result can be generalized. Suppose we were interested in the testable implications of preferences which are β-translation invariant, for some $\beta \geq 0$, $\beta \neq 0$. That is, we want to know whether for all x, y, we have $x \succeq y$ if and only if for all t, $x + t\beta \succeq y + t\beta$. Define the seminorm $\|x\|_1^\beta = \sum_i |\beta_i x_i|$. Then it is an easy exercise to verify that the testable implications of β-translation invariance are given by equation (4), replacing $\|\cdot\|_1$ with $\|\cdot\|_1^\beta$.

Remark: The test in (4) is related to cyclic monotonicity. This is similar to the test given in [3] for quasilinear preferences (and to a result in [16]).

We now turn our attention to maxmin preferences.

We say that a function $u : \mathbf{R}^S \to \mathbf{R}$ is linearly homogeneous if for all $x \in \mathbf{R}^S$ and all $\alpha > 0$, we have $u(\alpha x) = \alpha u(x)$.

Theorem 2: The following statements are equivalent:

1. Dataset D is rationalizable by a locally nonsatiated, homothetic and translation invariant preference.
2. Dataset D is rationalizable by a continuous, strictly increasing, linearly homogeneous and concave utility function satisfying the property that $u(x + (c, \dots, c)) = u(x) + c$.
3. Dataset D is risk-neutral maxmin-rationalizable.
4. For every k and l,

$$\frac{p^k}{\|p^k\|_1} \cdot x^k \leq \frac{p^l}{\|p^l\|_1} \cdot x^k.$$

The equivalence between (2) and (3) is due to [8]. Here we prove it through an application of Theorem 1.

It is interesting to note that, just as in Theorem 1, under the maintained hypotheses of risk aversion and monotonicity, uncertainty aversion has no content for behavior.

Remark: The rationalizing variational and maxmin preferences can be taken to imply "full support" priors. In the proof of Theorem 1, we shown that there is $\pi \in \Delta(S)$ satisfying $c(\pi) < +\infty$, which implies for all $s \in S$, $\pi_s > 0$. And in the proof of Theorem 2 we show that for each $\pi \in \Pi$ and all $s \in S$, $\pi_s > 0$.

CARA AND CRRA

The previous section considers translation invariance and homotheticity as general properties of preferences in choice under uncertainty. Here we focus on the case of subjective expected utility. So we consider models in which the agent has a single prior over states, and maximizes expected utility. The prior is unknown and must be inferred from her choices. Translation invariance gives rise to CARA preferences, and homotheticity to CRRA.

Theorem 3: A dataset D is CARA rationalizable if and only if there is $\alpha^* > 0$ such that (1) holds for all $k, k' \in K$ and $s, t \in S$; and CRRA rationalizable if and only if there is $\alpha^* \in (0, 1)$ such that (2) holds for all $k, k' \in K$ and $s, t \in S$.

$$\alpha^*(x_t^k - x_s^k + x_s^{k'} - x_t^{k'}) = \log\left(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}\right) \qquad [\mathbf{1}]$$

$$\alpha^* \log\left(\frac{x_t^k}{x_s^k}\frac{x_s^{k'}}{x_t^{k'}}\right) = \log\left(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}\right) \qquad [\mathbf{2}]$$

The conditions in Theorem 3 may look like existential conditions: essentially Afriat inequalities. Afriat inequalities are indeed the source of equations (1) and (2), as evidenced by the proof of Theorem 3, but note that the statements are equivalent to non-existential statements: Equation (1) says that when $(x_t^k - x_s^k + x_s^{k'} - x_t^{k'}) \neq 0$,

$$\frac{\log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}})}{(x_t^k - x_s^k + x_s^{k'} - x_t^{k'})}$$

is independent of k, t, k' and s; and that when $(x_t^k - x_s^k + x_s^{k'} - x_t^{k'}) = 0$ then $\log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}) = 0$. Similarly for equation (2).

It is worth pointing out that, except in the case when for all observations, all prices are equal, and consumption of all goods are equal, equation (1) can have only one solution. Hence, risk preferences are uniquely identified.

The next corollary also shows that beliefs are identified. Recall that a CARA utility is defined by a pair (a, π), with $a > 0$ and $\pi \in \Delta(S)$.

Corollary: If (a, π) and (a', π') define CARA utilities that rationalize D, then $(a, \pi) = (a', \pi')$. Furthermore, $a = a'$ coincide with the unique solution to (1). Similarly for CRRA rationalizability and (2).

RISK AVERSE MAXMIN WITH TWO STATES

Theorem 2 is about risk neutral maxmin. Here we turn to maxmin with risk aversion. In this section, we assume that there are two states (i.e., $S = \{1, 2\}$). A preference relation is *maxmin* if there is a closed and convex set $\Pi \subseteq \Delta(S)$, where for each $\pi \in \Pi$ and each $s \in S$, $\pi_s > 0$, and a concave utility $u : \mathbf{R}^S \to \mathbf{R}$ such that the utility function

$$\inf_{\pi\in\Pi} \sum_{s=1,2} \pi_s u(x_s)$$

represents $\succeq$. If a data set is rationalizable by a maxmin preference relation, we will say that the dataset set is *maxmin-rationalizable.*

Let K_0 be the set of all k such that $x^k_1 = x^k_2$. Let K_1 be the set of all k such that $x^k_1 < x^k_2$, and K_2 be the set of all k such that $x^k_1 > x^k_2$. Note that $K = K_0 \cup K_1 \cup K_2$.

Say that a sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ is *balanced* if each k appears as k_i (on the left of the pair) the same number of times it appears as k'_i (on the right).

Given a sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$, consider the following notation: Let $I_{l,s} = \{i : k_i \in K_l \text{ and } s_i = s\}$, $I'_{l,s} = \{i : k'_i \in K_l \text{ and } s'_i = s\}$, for $l = 0, 1, 2$ and $s = 1, 2$.

Axiom: Strong Axiom of Revealed Maxmin Expected Utility (SARMEU) For any balanced sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ in which

1. $x^{k_i}_{s_i} > x^{k'_i}_{s'_i}$ for all i;
2. $\#I_{0,1} + \#I_{1,1} - \#I'_{1,1} = \#I'_{0,1} + \#I'_{2,1} - \#I_{2,1} \le 0$

The product of prices satisfies that

$$\prod_{i=1}^{n} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \le 1. \qquad [3]$$

Theorem 4: A dataset is maxmin rationalizable if and only if it satisfies SARMEU.

[6] show that a stronger axiom, Strong Axiom of Revealed Subjective Expected Utility (SARSEU), characterizes rationalizability by subjective expected utility. Instead of condition (2) of SARMEU, SARSEU requires

$$\#I_{0,1} + \#I_{1,1} + \#I_{2,1} = \#I'_{0,1} + \#I'_{1,1} + \#I'_{2,1}. \qquad [4]$$

Theorem 4 is useful because it makes explicit what one would need to see in an experiment (with two states, a common setup in laboratory experiments) in order for choices to be consistent with maxmin utility, but inconsistent with subjective expected utility. For a dataset to be maxmin rationalizable, but inconsistent with subjective expected utility, it needs to contain a sequence in the conditions of SARSEU in which $\#I_{0,1}+\#I_{1,1}+\#I_{2,1} = \#I'_{0,1}+\#I'_{1,1}+\#I'_{2,1}$, but where $\#I_{0,1} + \#I_{1,1} - \#I'_{1,1} > 0$.

As we have emphasized, the result in Theorem 4 is for two states. There are two simplifications afforded by the assumption of two states, and the two are crucial in obtaining the theorem. The first is that with two states there are only two extreme priors to any set of priors. With the assumption that u is monotonic, one can know which of the two extremes is relevant to evaluate any given act.[4] The second simplification is a bit harder to see, but it comes from the fact that one can normalize the probability of one state to be one and only keep track of the probability of the other state. Then the property of being an extreme prior carries over to the probability of the state that is left "free."[5]

Type	Allocation	K	MEU
1a	Color 2 and 3	14	0.054
1b	Color 3 and 1	15	0.031
1c	Color 1 and 2	12	0.070
2a	Color 1 and (2 or 3)	9	0.078
2b	Color 2 and (1 or 3)	14	0.039
2c	Color 3 and (1 or 2)	12	0.054

TESTING MAXMIN

[11] study models of decision making under uncertainty using data from a laboratory experiment. 129 subjects are asked to allocate 50 experimental tokens between two states, states s or s'. Tokens allocated to each state have a value of a_s and $a_{s'}$. If a subject decides to allocate c_s tokens to state s, then he obtains a payment of $c_s \cdot a_s$ when state s realizes; and $(50 - c_s) \cdot a_{s'}$ when state s' realizes.

In each decision problem, each subject's decision is characterized by a triple $(a_s, a_{s'}, c_s)$, where c_s is the number of tokens she decides to allocate to state s. To map such decision to our notion of data, set prices to be $p_s = a_{s'}/a_s$ and $p_{s'} = 1$ (a normalization), and $I = 50 \cdot a_{s'}$. Then, we define consumptions (monetary amounts) as $x_s = c_s \cdot a_s$ and $x_{s'} = (50 - c_s) \cdot a_{s'}$.

In the experiment, there are three underling states: "color" 1, 2, and 3. But *only two states are relevant in each decision.* So we can test SARMEU. [11] used a Bingo Blower to decide a realization of a state. The Bingo Blower is a rectangular-shaped, glass-sided object in which many balls, whose color is either 1,2, or 3, are in continuous motion being moved by a wind from a fan in the base. A ball is drawn is from the Bingo Blower and the color of the ball determines the state.[6]In total, each subject thus completes 76 decision problems. There are two types of decision problems. Type 1 problems asked subjects to allocate tokens between two of the three colors, while type 2 problems asked them to make allocations between one of the three colors and the other two. There were 41 type 1 problems and 35 type 2 problems. For example, in type 1 problem, state $s = \{\text{color } 1\}$; state $s' = \{\text{color } 2\}$. In type 2 problem, state $s = \{\text{color } 1\}$; state $s' = \{\text{color } 2, \text{color } 3\}$.

One of the conclusions by [11] is that according to the Bayesian Information Criterion, the loss in predictive power in using SEU instead of generalizations of SEU is relatively small in magnitude. We test SARMEU for each individual subject and for each type of decision problem. The tests are based on linearized Afriat inequalities presented in the proofs of the theorems in [6] and in Lemma 1 of Theorem 4.

The table summarizes the results. Across six types of decision problems, we find that about 3% to 8% of the 129 subjects are MEU rational. Our result shows that MEU does not explain the subjects choices. This implies that SEU, a special case of MEU, does not explain the subjects choices either. One conclusion of our results is that decision theorists' efforts to account for experimental behavior does not seem to go very far in explaining the Hey-Pace data.

PROOFS

We provide the proofs of Theorem 1,2, and 3. We omit the proof of Theorem 4, which is similar to the proofs in [6].[7]

[4] This would also be true in the model of [17], whose uncertainty averse counterpart is equivalent to MEU in the case of two states.

[5] This can be seen in the proof of Lemma 1 when we go from $\bar{\pi} \ge \underline{\pi}$ to $\bar{\mu}_1 \ge \underline{\mu}_1$.

[6] The idea behind the use of a Bingo Blower was that subjects could not have sufficient information to calculate objective probabilities.

[7] The omitted proof is available in the authors' web page.

PROOF OF THEOREM 1. That [**3**] $\Longrightarrow$[**1**] is obvious. We shall first prove that [**1**] $\Longrightarrow$[**4**]

Suppose, towards a contradiction, D is a dataset satisfying [**1**] but not [**4**]. Then we have a cycle $\sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) < 0$. Let us without loss assume the sequence is $x^1, \ldots, x^M$ so as to avoid cumbersome notation. Let $Z = \sum_{l=1}^{M} \frac{p^l}{\|p^l\|_1} \cdot (x^{l+1} - x^l) < 0$.

Define a new sequence $(y^1, \ldots, y^M)$ inductively. Let $y^1 = x^1$, and let $y^k = x^k + (c^k, \ldots, c^k)$ where c^k is chosen so that $\frac{p^k}{\|p^k\|_1} \cdot (y^{k+1} - y^k) = \frac{Z}{M}$. Specifically, $c^1 = 0$ and

$$c^{k+1} = c^k + \frac{Z}{M} - \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k)$$

for $k = 1, \ldots, M-1$. Let $q^k = \frac{p^k}{\|p^k\|_1}$ and consider the dataset (q^k, y^k), $k = 1, \ldots M$.

The original dataset is rationalizable by some locally nonsatiated and translation invariant preference $\succeq$. It is easy to see that the same preference rationalizes the dataset (q^k, y^k). Indeed, if $q^k \cdot y^k \geq q^k \cdot y$ then $p^k \cdot x^k \geq p^k \cdot (y - (c^k, \ldots, c^k))$, by definition of y^k and q^k. So $x^k \succeq (y - (c^k, \ldots, c^k))$, and thus $y^k \succeq y$ by translation invariance of $\succeq$.

Observe that

$$\begin{aligned}
&\sum_{k=1}^{M-1} q^k \cdot (y^{k+1} - y^k) + q^M \cdot (y^1 - y^M) \\
= \;& \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) \\
&\quad + \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot ((c^{k+1}, \ldots, c^{k+1}) - (c^k, \ldots, c^k)) \\
= \;& \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) + \sum_{k=1}^{M} \frac{(\sum_{s \in S} p^k_s)(c^{k+1} - c^k)}{\|p^k\|_1} \\
= \;& \sum_{k=1}^{M} \frac{p^k}{\|p^k\|_1} \cdot (x^{k+1} - x^k) \quad (\because \|p^k\|_1 = \sum_{s \in S} p^k_s) \\
= \;& Z \quad (\because \text{Definition of } Z),
\end{aligned}$$

and that $q^k \cdot (y^{k+1} - y^k) = Z/M$ for $k = 1, \ldots, M-1$. Therefore, $q^M \cdot (y^1 - y^M) = Z/M$. In particular, $q^k \cdot (y^{k+1} - y^k) = Z/M < 0$ for $k = 1, \ldots, M \pmod{M}$. Thus $y^k \succ y^{k+1}$ as (q^k, y^k) is rationalizable by $\succeq$ and $\succeq$ is locally nonsatiated. This contradicts the transitivity of $\succeq$.

Now we show that [**4**] $\Longrightarrow$[**2**]. Let $x \in \mathbf{R}^S$. Let Σ_x be the set of all subsequences $\{k_l\}_{l=1}^M \subset \{1, \ldots, K\}$ for which $k_1 = 1$ and define $x^{k_{M+1}} = x$. By [**4**], if $\{k_l\}_{l=1}^M \in \Sigma_x$ has a cycle (meaning that $k_l = k_{l'}$ for $l, l' \in \{1, \ldots, M\}$ with $l \neq l'$), then there is a shorter sequence $\{k_j\}_{j=1}^{M'} \in \Sigma_x$ with

$$\sum_{j=1}^{M'} \frac{p^{k_j}}{\|p^{k_j}\|_1} \cdot (x^{k_{j+1}} - x^{k_j}) \leq \sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}).$$

Therefore, $u(x) = \inf\{\sum_{l=1}^{M} \frac{p^{k_l}}{\|p^{k_l}\|_1} \cdot (x^{k_{l+1}} - x^{k_l}) : \{k_l\}_{l=1}^M \in \Sigma_x\}$ is well defined, as the infimum can be taken over a finite set.

That $u : \mathbf{R}^S \to \mathbf{R}$ defined in this fashion is concave, strictly increasing and continuous is immediate. To see that it rationalizes the data, suppose that $p^k \cdot x^l \leq p^k \cdot x^k$. Then $\frac{p^k}{\|p^k\|_1} \cdot x^l \leq \frac{p^k}{\|p^k\|_1} \cdot x^k$. It is clear then by definition that $u(x^l) \leq u(x^k) + \frac{p^k}{\|p^k\|_1} \cdot (x^l - x^k) \leq u(x^k)$.

Finally, to show that $u(x + (c, \ldots, c)) = u(x) + c$, note that for any p^k, we have $\frac{p^k}{\|p^k\|_1} \cdot (x + (c, \ldots, c)) = c + \frac{p^k}{\|p^k\|_1} \cdot x$. The result then follows by construction.

We end the proof by showing that [**2**] $\Longrightarrow$[**3**] Let $u : \mathbf{R}^S \to \mathbf{R}$ be as in the statement of [**2**]. Define the concave conjugate of u by

$$\begin{aligned}
f(\pi) &= \inf\{\pi \cdot x - u(x) : x \in \mathbf{R}^S\} \\
&= \inf\{\pi \cdot x + c\pi \cdot \mathbf{1} - u(x) - c : x \in \mathbf{R}^S, c \in \mathbf{R}\} \\
&= \inf\{\pi \cdot x - c(1 - \pi \cdot \mathbf{1}) - u(x) : x \in \mathbf{R}^S, c \in \mathbf{R}\},
\end{aligned}$$

where the second equality uses that $u(x+(c, \ldots, c)) = u(x)+c$. Now note that $f(\pi) = -\infty$ if $(1 - \pi \cdot \mathbf{1}) \neq 0$. Note also that the monotonicity of u implies that $f(\pi) = -\infty$ if there is s such that $\pi_S < 0$. One can also show that there is $\pi \in \Delta(S)$ for which $f(\pi) \in \mathbf{R}$.[8] Finally, observe that by strict monotonicity, if there is $s \in S$ for which $\pi_s = 0$, then $f(\pi) = -\infty$. Hence we can consider the domain of f to be a subset of $\Delta(S)$. Moreover, $f(\pi) < +\infty$ implies for all $s \in S$, $\pi_s > 0$.

Now since u is continuous, it is a standard application of the separating hyperplane theorem to establish that $u(x) = \inf_{\pi \in \Delta(S)} \pi \cdot x - f(\pi)$. Since u rationalizes the dataset, the dataset is variational rationalizable.

PROOF OF THEOREM 2. It is obvious that [**3**] $\Longrightarrow$[**2**] and that [**2**] $\Longrightarrow$[**1**]. Hence, to show the theorem, it suffices to show that [**4**] implies [**3**] and that [**1**] implies [**4**].

For a dataset D, let $\pi^k = \frac{p^k}{\|p^k\|_1}$. It is easy to see that [**4**] $\Longrightarrow$[**3**]. Let Π be the convex hull of $\{\pi^k : k = 1, \ldots, K\}$. Then it is immediate that $u(x) = \min_{\pi \in \Pi} \pi \cdot x$ rationalizes D. Moreover, for each $\pi \in \Pi$ and all $s \in S$, $\pi_s > 0$ because $\pi^k_s > 0$ for all $s \in S$ and $k \in K$.

We prove that [**1**] $\Longrightarrow$[**4**]. Suppose that D satisfies [**1**] but not [**4**]. Then there are k and l for which $\pi^l \cdot x^k < \pi^k \cdot x^k$. Let $\succeq$ be a preference relation as stated in [**1**]. By homotheticity of $\succeq$, for any scalar $\theta > 0$, $\succeq$ rationalizes the data $D' \equiv \{(x^j, \pi^j) : j = 1, \ldots, K\} \cup \{(\theta x^l, \pi^l)\}$. To see this, observe that if $\pi^l \cdot x \leq \pi^l \cdot \theta x^l$, then $\pi^l \cdot \theta^{-1} x \leq \pi^l x^l$, so that $x^l \succeq \theta^{-1} \cdot x$, and by homogeneity, $\theta x^l \succeq x$. Now, for $\theta > 0$ sufficiently small, $\pi^l \cdot x^k < \pi^k \cdot x^k$ implies that

$$x^k \cdot (\pi^l - \pi^k) + \theta x^l \cdot (\pi^k - \pi^l) < 0.$$

So either $x^k \cdot (\pi^l - \pi^k) < 0$ or $\theta x^l \cdot (\pi^k - \pi^l) < 0$. Then the dataset D' violates [**4**] in Theorem 1, contradicting the fact that it is rationalized by $\succeq$, which is assumed to be translation invariant.

PROOF OF THEOREM 3. The idea in the proof is to solve the first-order conditions for the unknown terms. Consider first the case of CARA. Let $\pi \in \Delta(S)$ and $\alpha > 0$ rationalize D. Then we know that x^k maximizes $\sum_s \pi_s (-\exp(-\alpha x_s))$ subject to $p^k \cdot x \leq p^k \cdot x^k$. By considering the Lagrangean

[8] For example, take π to support $\{z \in \mathbf{R}^S : u(z) \geq 0\}$ at 0. We claim that $f(\pi) \geq 0$. Suppose by means of contradiction that there is $x \in \mathbf{R}^S$ for which $\pi \cdot x < u(x)$. Observe that π supports $\{z \in \mathbf{R}^S : u(z) \geq \pi \cdot x\}$ at the act y which returns $\pi \cdot x$ in each state. Observe that $u(z) > \pi \cdot x$ implies $\pi \cdot z > \pi \cdot x$, by continuity of u and definition of the supporting hyperplane; that is, $\{z \in \mathbf{R}^S : u(z) \geq \pi \cdot x\} \subseteq \{z \in \mathbf{R}^S : \pi \cdot z \geq \pi \cdot x\}$ implies $\{z \in \mathbf{R}^S : u(z) > \pi \cdot x\} \subseteq \{z \in \mathbf{R}^S : \pi \cdot z > \pi \cdot x\}$ as the latter sets are the interiors of the former. Therefore, if $u(x) > \pi \cdot x$, we conclude $\pi \cdot x > \pi \cdot x$, a contradiction.

and the first order conditions, we may conclude that for every $s, t \in S$ and every $k \in \{1, \dots, K\}$, we have

$$\frac{\pi_s \exp(-\alpha x_s^k)}{p_s^k} = \frac{\pi_t \exp(-\alpha x_t^k)}{p_t^k}.$$

Conclude that $\frac{p_s^k \pi_t}{p_t^k \pi_s} = \exp(-\alpha(x_s^k - x_t^k))$. By taking logs, the system becomes:

$$\log(\pi_s) - \log(\pi_t) + \alpha(x_t^k - x_s^k) = \log(p_s^k) - \log(p_t^k). \quad [\mathbf{5}]$$

In the case of CRRA, the existence of a rationalizing π and parameter α imply a first-order condition of the form

$$\log(\pi_s) - \log(\pi_t) + \alpha \log(x_t^k / x_s^k) = \log(p_s^k) - \log(p_t^k). \quad [\mathbf{6}]$$

We can denote $\log(\pi_s)$ by z_s in equations [**5**] and [**6**]. Thus we obtain that D is rationalizable if and only if there exist $z_s \in \mathbf{R}$ and $\alpha > 0$ such that the following equation is solved for all s, t, k with $s \neq t$:

$$z_s - z_t + \alpha(y_t^k - y_s^k) = \log(p_s^k) - \log(p_t^k),$$

where $y_t^k = x_t^k$ for CARA rationalizability, and $y_t^k = \log x_t^k$ for CRRA rationalizability.

Now the necessity of the axioms is obvious. Let $k \neq k'$, then

$$\alpha(y_t^k - y_s^k) - \log(p_s^k/p_t^k) = z_s - z_t = \alpha(y_t^{k'} - y_s^{k'}) - \log(p_s^{k'}/p_t^{k'})$$

for any s and t. Thus

$$\alpha(y_t^k - y_s^k - y_t^{k'} + y_s^{k'}) = \log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}).$$

So [**1**] is satisfied for the case of CARA rationalizability, and [**2**] is satisfied for the case of CRRA rationalizability.

To prove sufficiency, let

$$d^p(s,t,k) = \log(p_s^k/p_t^k)$$
$$d^x(s,t,k) = y_s^k - y_t^k.$$

Let α^* be such that for all k, k', s, s' and t,

$$\alpha^*(y_t^k - y_s^k - y_t^{k'} + y_s^{k'}) = \log(\frac{p_s^k}{p_t^k}\frac{p_t^{k'}}{p_s^{k'}}).$$

Then in particular, for all k, k', s, s' and t,

$$d^p(s,t,k) + \alpha^* d^x(s,t,k) + d^p(t,s,k') + \alpha^* d^x(t,s,k') = 0. \quad [\mathbf{7}]$$

Note also that

$$\begin{aligned} d^p(s,t,k) + d^p(t,s',k) + d^p(s',s,k) \\ + \alpha^*(d^x(s,t,k) + d^x(t,s',k) + d^x(s',s,k)) = 0. \end{aligned} \quad [\mathbf{8}]$$

Fix $s_0 \in S$ and let $z_{s_0} \in \mathbf{R}$ be arbitrary. For any $s \in S$, define z_s by

$$z_s = z_{s_0} + \alpha^* d^x(s_0, s, k) + d^p(s, s_0, k),$$

for some k. In fact, by equation [**7**], this definition is independent of k because $d^p(s, s_0, k) + \alpha^* d^x(s, s_0, k) = d^p(s, s_0, k') + \alpha^* d^x(s, s_0, k')$.

Given this definition, note that

$$\begin{aligned} z_s - z_t &= \alpha^*(d^x(s_0,s,k) - d^x(s_0,t,k)) + d^p(s,s_0,k) - d^p(t,s_0,k) \\ &= \alpha^*(d^x(s_0,s,k) - d^x(s_0,t,k)) + d^p(s,s_0,k) - d^p(t,s_0,k) \\ &+ d^p(s,t,k) + d^p(t,s_0,k) + d^p(s_0,s,k) \\ &+ \alpha^*(d^x(s,t,k) + d^x(t,s_0,k) + d^x(s_0,s,k)) \\ &= d^p(s,t,k) + \alpha^* d^x(s,t,k), \end{aligned}$$

where the second equality uses equation [**8**].

Hence, with the constructed $(z_t)_{t \in S}$ we have

$$z_s - z_t + \alpha^*(y_t^k - y_s^k) = \log(p_s^k/p_t^k),$$

for all s, t, and k. The first-order conditions for rationalizability are therefore satisfied.

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1. Ahn, D., S. Choi, D. Gale, and S. Kariv (2014): Estimating ambiguity aversion in a portfolio choice experiment, Quantitative Economics.
2. Bayer, R., S. Bose, M. Polisson, and L. Renou (2012): Ambiguity Revealed, Mimeo, University of Essex.
3. Brown, D. J. and C. Calsamiglia (2007): The Nonparametric Approach to Applied Welfare Analysis, Economic Theory, 31, 183–188.
4. Chambers, C. P., F. Echenique, and E. Shmaya (2014): The axiomatic structure of empirical content, American Economic Review, 104, 2303–2319.
5. Cerreia-Vioglio, S., Maccheroni F. Marinaccia, M., and L. Montrucchio (2013): Classical subjective expected utility, PNAS 110 (17) 6754-6759.
6. Echenique, F. and K. Saito (2015): Savage in the market, Econometrica, 83(4), 1467-1495.
7. Ellsberg, D. (1961): Risk, ambiguity, and the Savage axioms, Quarterly Journal of Economics, 75, 643–669.
8. Gilboa, I. and D. Schmeidler (1989): Maxmin expected utility with non-unique prior, Journal of mathematical economics, 18, 141–153.
9. Green, R. C., and Srivastava, S. (1986). Expected utility maximization and demand behavior. Journal of Economic Theory, 38(2), 313–323.
10. Hansen, L. P. and T. J. Sargent (2008): Robustness, Princeton university press.
11. Hey, J. and N. Pace (2014): The explanatory and predictive power of non two-stage-probability theories of decision making under ambiguity, Journal of Risk and Uncertainty, 49, 1–29.
12. Kubler, F., L. Selden, and X. Wei (2014): Asset Demand Based Tests of Expected Utility Maximization, American Economic Review, forthcoming.
13. Maccheroni, F., M. Marinacci, and A. Rustichini (2006): Ambiguity Aversion, Robustness, and the Variational Representation of Preferences, Econometrica, 74, 1447–1498.
14. Cerreia-Vioglio S., Maccheroni, F., Marinacci, M., and L. Montrucchio (2011): Uncertainty Averse Preferences Journal of Economic Theory 146, 1275-1330.
15. Polisson, M. and J. Quah (2013): Revealed preference tests under risk and uncertainty, Working Paper 13/24 University of Leicester.
16. Rockafellar, R. T. (1997): Convex analysis, Princeton university press.
17. Schmeidler, D. (1989): Subjective probability and expected utility without additivity, Econometrica, 57, 571–587.
18. Stoer, J. and C. Witzgall (1970): Convexity and optimization in finite dimensions, Springer-Verlag Berlin.
19. Strzalecki, T. (2011): Axiomatic foundations of multiplier preferences, Econometrica, 79, 47–73.
20. Varian, H. R. (1988): Estimating risk aversion from Arrow-Debreu portfolio choice, Econometrica, 43, 973–979.